Classification of Quaternary \([21s+17,3]\) Optimal Self-orthogonal Codes

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Abstract

The classification of quaternary \([21s+t,3,d]\) codes with \(d \geq 16s\) and without zero coordinates is reduced to the classification of quaternary \([21c(3,s,t)+t,k,d]\) code for \(s \geq 1\) and \(0 \leq t \leq 20\), where \(c(3,s,t) \leq \min\{3s,t\}\) is a function of \(3s\) and \(t\). Quaternary optimal Hermitian self-orthogonal codes are characterized by systems of linear equations. Based on these two results, the complete classification of \([21s+17,s]\) optimal self-orthogonal codes for \(s \geq 1\) is obtained, and the generator matrices and weight polynomials of these 3-dimensional optimal self-orthogonal codes are also given. All these codes meeting the Griesmer bound.

Keywords: Quaternary linear code, Self-orthogonal code, Optimal code, Griesmer bound.

1. Introduction

Since the pioneer work of MacWilliams et. al. in [1] in 1978, people paid much attention on quaternary Hermitian self-dual codes—a subclass of self-orthogonal codes (SO codes, for short) over the quaternary field \(F_4\), and a vast number of paper have been devoted to the study of quaternary self-dual codes, see the excellent survey of [2] and [3] for an overview of these results and the references therein.

Recently, with the development of quantum error-correcting codes and the classification of self-dual codes, people begin to study optimal \([n,k]\) SO codes over \(F_4\) with \(n \geq 40\) and dimension less than 10, and give complete classification of 3-dimensional optimal SO codes in [4-5]. In [6], Bouykliev et. al. studied the classification of binary optimal SO codes of length \(n \leq 40\) and dimension less than 10, and gave complete classification of 3-dimensional optimal SO codes in [7]. Bouykliev et. al. studied the classification of optimal SO codes of length \(n \leq 29\) and dimension less than 7 over \(F_3\) and \(F_4\), and give complete classification of \([n,3]\) optimal self-orthogonal codes over \(F_4\) for \(6 \leq n \leq 29\). Their results of optimal SO codes over \(F_4\) show that some optimal SO codes meeting the Griesmer bound are unique under equivalence. In [8], we studied the classification of binary \([15s+t,4]\) optimal SO codes for \(s \geq 1\) and \(t \in \{1,2,6,7,8,9,13,14\}\). In [9], Ruihu Li proved that \([21s+t,3]\) optimal self-orthogonal codes over \(F_4\) with \(s \geq 1\) and \(t \in \{0,16\}, 19\) meeting the Griesmer bound are unique under equivalence.

In this paper, we give complete classification of \([21s+17,3]\) optimal self-orthogonal codes over \(F_4\) with \(s \geq 1\).

The paper is arranged as follows. First, we give some notations and make some preparation in this section. In section 2, we give the relations of quaternary SO codes and some systems of linear equations, and explain how to determine the \([21s+t,3]\) optimal SO codes. In section 3, we give the classification of \([21s+17,3]\) optimal SO codes for \(s \geq 1\).

Let \(F_4 = \{0,1,\omega, \omega^3\}\) be the Galois field with four elements such that \(\omega = 1 + \omega = \omega^3\), \(\omega^4 = 1\), and the conjugation is defined by \(\Gamma(x) = x^3\). Let \(F_4^n\) be the \(n\)-dimensional row vector space over the quaternary field \(F_4\). A \(k\)-dimensional subspace \(C\) of \(F_4^n\) is called a quaternary linear \([n,k]\) code.
The Hermitian inner product of $v \in F_4^n$ is defined as:

$$(u, v)_h = u^T v = \sum_{i=1}^{n} u_i v_i^*.$$ 

The Hermitian dual code $C^\perp$ of $C$ is defined as $C^\perp = \{u \in F_4^n | (u, v)_h = 0 \text{ for all } v \in C\}$. A code $C$ is self-orthogonal if $C \subseteq C^\perp$, and self-dual if $C = C^\perp$. A quaternary code is SO if and only if $k$ is even, i.e., the weights of all codewords in $C$ are even, see [4] and [10].

Let $A$ be the generator matrix of the Simplex code $C_3 = [21, 3, 16]$ and $G$ be a generator matrix of the Simplex code $C_1 = [n, k]$ SO code $C$ is called optimal if it has the highest weight among all $[n, k]$ SO codes.

Two quaternary codes $C$ and $C'$ are equivalent if one can be obtained from the other by permuting the coordinates, multiply some coordinates by non zero elements, or conjugate all the coordinates. If two matrices $G$ and $G'$ generate equivalent codes, we denote them as $G \equiv G'$.

We use $I_n = (1,1,1)_n$ and $0_n = (0,0,...,0)_n$ to denote the all-ones vector and the zero vector of length $n$, respectively. And use $iG = (G, G, ..., G)$ to denote the juxtaposition of $i$ copies of $G$ for given matrix $G'$.

### 2. Quaternary Self-orthogonal Codes and Systems of Linear Equations

In order to present our main results, we need some preparations.

A row (or column) vector is monic if its first non zero coordinate is 1. Denote $N_4 = \frac{4^k - 1}{3}$, then there are $N_4$ monic vectors of dimension $k$.

**Definition 2.1** For a linear code $C = [n, k, d]$ with generator matrix $G$, a matrix whose rows are formed by all monic vectors of $C$ is called a projective matrix of $C$ and denoted as $P(C)$ or $P(G)$. Let $P(G) = (\beta_1^T, \beta_2^T, ..., \beta_n^T)^T$ and $W(C) = (w(\beta_1), w(\beta_2), ..., w(\beta_n)) = (x_1, x_2, ..., x_n)$, $W(C)$ is called a projective weight vector of $C$, and $x_1, x_2, ..., x_n$ are called projective weights of $C$.

If $P = (a_{ij})$ is a matrix over $F_4$, its projection $(P)_r = (b_{ij})$ is a binary matrix, where $b_{ij} = 0$ if $a_{ij} = 0$ and $b_{ij} = 1$ otherwise.

Let $\alpha_1 = (0,1,0)^T$, $\alpha_2 = (1,0,0)^T$, $\alpha_3 = (1,1,0)^T$, $\alpha_4 = (1,\omega,0)^T$, $\alpha_5 = (1,\sigma,0)^T$, $\alpha_6 = (0,0,1)^T$, $\alpha_7 = (0,1,1)^T$, $\alpha_8 = (1,0,1)^T$, $\alpha_9 = (1,1,1)^T$, $\alpha_{10} = (1,\omega,1)^T$, $\alpha_{11} = (1,\sigma,1)^T$, $\alpha_{12} = (0,1,\omega)^T$, $\alpha_{13} = (1,0,\omega)^T$, $\alpha_{14} = (1,1,\omega)^T$, $\alpha_{15} = (1,\omega,\omega)^T$, $\alpha_{16} = (1,\sigma,\omega)^T$, $\alpha_{17} = (0,1,\sigma)^T$, $\alpha_{18} = (1,0,\sigma)^T$, $\alpha_{19} = (1,1,\sigma)^T$, $\alpha_{20} = (1,\omega,\sigma)^T$, and $\alpha_{21} = (1,\sigma,\sigma)^T$. Then all the monic vectors of dimension 3 are these 21 vectors. Fix $G_3 = (\alpha_1, \alpha_2, ..., \alpha_{21})$, then $G_3$ is a generator matrix of the Simplex code $C_3 = [21, 3, 16]$.

Let $P_2$ and $A_2 = (P_2)_r$ as

$$P_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \bar{\omega} \\ 1 & 1 & 0 & \bar{\omega} & \omega \\ 1 & \omega & \bar{\omega} & 0 & 1 \\ 1 & \bar{\omega} & \omega & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}. $$
And, let $J_5$ be the $5 \times 5$ all one matrix, denote $P_2 = P_{2,0}$, $P_{2,1} = P_2 + J_5$, $P_{2,2} = P_2 + \alpha J_5$ and $P_{2,3} = P_2 + \alpha J_5$, their projections are denoted as $A_2 = (P_{2,0})_P$ and $A_{2,i} = (P_{2,i})_P$, $1 \leq i \leq 3$, respectively.

From the generator matrix $G_3$ of $C_3$, we have $P_3 = P(C_3)$ and $A_3 = (P_3)_P$ as follow:

$$P_3 = \begin{pmatrix} P_2 & 0 & P_2 & P_2 \\ 0 & 1 & I_5 & \omega I_5 & \bar{\omega} I_5 \\ P_2 & \omega I_5 & P_{2,1} & P_{2,2} & P_{2,3} \\ P_2 & \bar{\omega} I_5 & P_{2,2} & P_{2,3} & P_{2,4} & P_{2,2} \end{pmatrix}, A_3 = \begin{pmatrix} A_2 & 0 & A_2 & A_2 \\ 0 & 1 & I_5 & \omega I_5 & \bar{\omega} I_5 \\ A_2 & \omega I_5 & A_{2,1} & A_{2,2} & A_{2,3} \\ A_2 & \bar{\omega} I_5 & A_{2,2} & A_{2,3} & A_{2,4} & A_{2,2} \end{pmatrix}.$$

Now, we can use systems of linear equations to characterize SO codes of given minimum distance as in [8].

From the definition of equivalence of quaternary linear codes, we can suppose an $[n,3]$ code $C$ with generator matrix $G$, where the columns of $G$ are all monic. If the columns of $G$ have $l_i$ copies of $\alpha_i$ for $1 \leq i \leq 21$, we denote $G$ as $G = (l_i \alpha_1, l_2 \alpha_2, \ldots, l_{21} \alpha_{21})$ for short and call such $G$ in order form, and call $L_G = (l_1, l_2, \ldots, l_{21})$ the define vector of $G$ and $C$ the code corresponding to $L_G$. Let $Y^T = A_3 L_G^T$, where $Y = (y_1, y_2, \ldots, y_{21})$. Then $Y = (y_1, y_2, \ldots, y_{21})$ is a projective weight vector of $C$. Since $GL(3, F_4)$ acts double transitively on $\{\alpha_1, \ldots, \alpha_{21}\}$, hence, in the following, we can assume $l_1 \geq l_2 \geq \cdots \geq l_i$, $i = 3, 4, \ldots, 21$. Let $j \geq r_j$, $j = 7, \ldots, 21$ as in [6], and without further explain.

Let $C$ be an $[21s + t, 3, 16s + d_t]$ SO code with generator matrix $G$ and define vector $L = (l_1, l_2, \ldots, l_{21})$, where $s \geq 1$, $1 \leq t \leq 20$. Using the Griesmer bound, one can deduce an upper bound of $d_t$, hence the projective weight vector of $C$ can be assumed as $Y = (y_1, y_2, \ldots, y_{21})$, where $y_i = 16s + d_t + 2\lambda_i$ and $\lambda_i \geq 0$ for $1 \leq i \leq 21$. Thus, we have the following equations.

$$A_k \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_{21} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{21} \end{pmatrix} \quad \text{(1)}$$

Then, for given $Y$, an $[21s + t, k_s, 16s + d_t]$ SO code with projective weight vector $Y$ exist if and only if the above linear equations (1) has non-negative integer solution.

Let $L'_G = L_G - s1_{21}$. Then, from $Y^T = A_3 L'^T$, one can deduce that $2ld_t + 2(\lambda_1 + \lambda_2 + \lambda_{21}) = 4^{1-s} (l'_1 + l'_2 + \ldots + l'_{21})$, thus $2(\lambda_1 + \lambda_2 + \lambda_{21}) = 4^{1-s} t - 2ld_t$. Let $\Lambda = (\lambda_1, \lambda_2, \lambda_{21})$, $d_t = 2a_t$, $\delta_t = \lambda_1 + \lambda_2 + \lambda_{21}$, then an $[21s + t, 3, 4^{1-s} + d_t]$ SO code with projective weight vector $Y$ exist if and only if the following system of linear equations have integer solution $L'_G$ such that each $s + l'_i$ is non-negative.
The fifteen non-equivalent optimal SO codes is based on the following lemma given in [9], its proof is similar to the proof of Theorem 1.1 given by [8].

Lemma 2.1 Suppose, \( s \geq 1, \ 0 \leq t \leq 20 \) and \( n = 21s + t \). Then, every \([n, 3, d]\) quaternary code \( C \) with \( d \geq 16s \) and without zero coordinates is equivalent to a code with generator matrix \( G = ((s - c(3, s, t))G_3, H) \), where \( c(3, s, t) \leq \min \frac{s}{3}, 3t \) is a function of \( s \), \( 3 \), and \( t \). And \( H \) has \( 21c(3, s, t) + t \) columns.

According to this lemma, the classification of \([n, 3]\) optimal codes is changed into determining \( c(3, s, t) \), and can be reduced to the classification of \([m + t, 3]\) optimal SO codes for \( m \leq 21c(3, s, t) \).

For given \( s \geq 1 \) and \( 1 \leq t \leq 20 \). Denote the set of all the generator matrix \( G \) (determined above) of \([21s + t, 3]\) optimal SO codes as \( G[21s + t, 3] \), and let \( D[21s + t, 3] = \{ L_G \} \) be the define vectors of \( G, G \in G[21s + t, 3] \). Then \( c(3, s, t) = -\min_{i \in s \leq 21} \{ |L_i - s| L_G \in D[21s + t, 3] \} \). Thus, this explains how to determine the generator matrices of all \([n, 3]\) optimal SO codes and how one can determine \( c(3, s, t) \).

### 3. Results of Classification

In this section, we will use the results of the above two sections to study the classification of optimal \([n, 3]\) SO codes for \( n = 21s + 17, \ s \geq 1 \). Let \( N(n, 3) \) be the number of non-equivalent optimal \([n, 3]\) SO codes, and \( N_0(n, 3) \) and \( N_1(n, 3) \) be the number of non-equivalent optimal \([n, 3]\) SO codes with zero coordinates and without zero coordinates, respectively.

A \([21s + 17, 3]\) optimal SO code has minimum distance \( 16s + 12 \). We give our classification results as follow.

**Theorem 3.1** If \( n = 21s + 17, s \geq 1 \), then the following holds:

1. If \( s = 1 \), then \( N_1(38, 3) = 15 \). The fifteen non-equivalent optimal \([38, 3, 28]\) SO codes without zero coordinates have generator matrices \( G_{38,i} (1 \leq i \leq 15) \), and their weight polynomials are \( W_{38,i} \) as follow:

\[
G_{38,i} = (G_3, G_{0,j}) (1 \leq i \leq 4)
\]

Let \( G_{0,1} = \left\{ 2\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_8, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{19}, \alpha_{20}, \alpha_{21} \right\} \),
\[
G_{0,2} = \left\{ 2\alpha_1, 2\alpha_2, \alpha_5, \alpha_6, \alpha_8, \alpha_9, \alpha_{12}, \alpha_{13}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{19}, \alpha_{20}, \alpha_{21} \right\} \),
\[
G_{0,3} = \left\{ 2\alpha_1, 2\alpha_2, \alpha_4, 2\alpha_6, \alpha_9, \alpha_{10}, \alpha_{12}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{18}, \alpha_{19}, \alpha_{21} \right\} \),
\[
G_{0,4} = \left\{ 2\alpha_1, 2\alpha_2, \alpha_5, 2\alpha_6, \alpha_8, 2\alpha_{10}, \alpha_{12}, 2\alpha_{14}, \alpha_{15}, \alpha_{19}, 2\alpha_{21} \right\} \),
\[
G_{0,5} = \left\{ 2\alpha_1, \cdots, 2\alpha_6, 2\alpha_8, \cdots, 2\alpha_{16}, 2\alpha_{18}, \cdots, 2\alpha_{21} \right\} \),
\[
G_{0,6} = \left\{ 3\alpha_1, 3\alpha_2, 2\alpha_4, 2\alpha_5, 2\alpha_6, \alpha_7, \alpha_8, 2\alpha_9, \cdots, 2\alpha_{13}, \alpha_{14}, 2\alpha_{15}, \cdots, 2\alpha_{18}, \alpha_{19}, 2\alpha_{20}, 2\alpha_{21} \right\} \),
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$$G_{1,7} = \{3\alpha_1, 3\alpha_2, \alpha_3, 3\alpha_4, 2\alpha_5, \alpha_6, \alpha_7, 2\alpha_9, \alpha_{10}, 2\alpha_{11}, \alpha_{12}, 2\alpha_{14}, 3\alpha_{15}, 3\alpha_{16}, 2\alpha_{17}, \alpha_{18}, 2\alpha_{19}, \alpha_{20}, 2\alpha_{21}\},$$

$$G_{1,8} = \{3\alpha_1, 3\alpha_2, \alpha_3, 3\alpha_4, 2\alpha_5, 3\alpha_6, \alpha_7, 2\alpha_9, \alpha_{10}, 2\alpha_{11}, \alpha_{12}, 2\alpha_{13}, 3\alpha_{14}, 3\alpha_{15}, 3\alpha_{16}, 2\alpha_{17}, \alpha_{18}\},$$

$$G_{1,9} = \{3\alpha_1, 3\alpha_2, 2\alpha_3, \alpha_4, 2\alpha_5, 3\alpha_6, \alpha_7, 2\alpha_{10}, \alpha_{11}, 2\alpha_{12}, \alpha_{13}, 3\alpha_{14}, \alpha_{15}, 3\alpha_{16}, \alpha_{17}, \alpha_{18}\},$$

$$2\alpha_{19}, \alpha_{20}, \alpha_{21}\},$$

$$G_{1,10} = \{3\alpha_1, 3\alpha_2, 2\alpha_3, \alpha_4, 2\alpha_5, 3\alpha_6, \alpha_7, 2\alpha_9, \alpha_{10}, \alpha_{11}, 2\alpha_{12}, \alpha_{13}, 3\alpha_{14}, 3\alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{18}\},$$

$$2\alpha_{19}, \alpha_{20}, \alpha_{21}\},$$

$$W_{38,1} = 1 + 33y^{28} + 30y^{30}, W_{38,2} = 1 + 36y^{28} + 24y^{30} + 3y^{32},$$

$$W_{38,3} = 1 + 39y^{28} + 18y^{30} + 6y^{32}, W_{38,4} = 1 + 48y^{28} + 15y^{32},$$

$$W_{38,5} = 1 + 36y^{28} + 24y^{30} + 3y^{32}, W_{38,6} = 1 + 39y^{28} + 18y^{30} + 6y^{32},$$

$$W_{38,7} = 1 + 39y^{28} + 21y^{30} + 3y^{34}, W_{38,8} = 1 + 42y^{28} + 15y^{30} + 3y^{32} + 3y^{34},$$

$$W_{38,9} = 1 + 42y^{28} + 18y^{30} + 3y^{36}, W_{38,10} = 1 + 45y^{28} + 12y^{30} + 6y^{34},$$

$$W_{38,11} = 1 + 45y^{28} + 15y^{30} + 3y^{36}, W_{38,12} = 1 + 48y^{28} + 15y^{32},$$

$$W_{38,13} = 1 + 48y^{28} + 15y^{32}, W_{38,14} = 1 + 51y^{28} + 9y^{32} + 3y^{36},$$

$$W_{38,15} = 1 + 51y^{28} + 9y^{32} + 3y^{36},$$

(2) If $s = 2$, then $N_1(59,3) = 19$. The 19 non-equivalent optimal $[59, 3, 44]$ SO codes without zero coordinates have generate matrices $G_{2,i}$ ($1 \leq i \leq 19$), and their weight polynomials are $W_{59,i}$ as follow:

$$G_{2,1} = (G_{3,1}, G_{1,1}), W_{59,1} = 1 + y^{16}(W_{38,1} - 1), (1 \leq i \leq 15)$$

$$G_{2,16} = \{4\alpha_1, 4\alpha_2, 3\alpha_3, 4\alpha_4, 3\alpha_5, \alpha_6, 2\alpha_7, 3\alpha_8, \alpha_9, \alpha_{10}, 2\alpha_{11}, 3\alpha_{12}, 4\alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{18}, \alpha_{19}, 2\alpha_{20}, 2\alpha_{21}\},$$

$$G_{2,17} = \{4\alpha_1, 4\alpha_2, \alpha_3, 3\alpha_4, 4\alpha_5, \alpha_6, 2\alpha_7, 4\alpha_8, \alpha_9, 3\alpha_{10}, 2\alpha_{11}, 4\alpha_{12}, 3\alpha_{13}, \alpha_{14}, 3\alpha_{15}, 2\alpha_{16}, \alpha_{17}, \alpha_{18}, 2\alpha_{19}, 2\alpha_{20}, 4\alpha_{21}\},$$

$$G_{2,18} = \{4\alpha_1, 4\alpha_2, \alpha_3, 2\alpha_4, 4\alpha_5, 4\alpha_6, 2\alpha_8, 4\alpha_9, 4\alpha_{10}, \alpha_{11}, 4\alpha_{12}, 2\alpha_{13}, 4\alpha_{14}, 3\alpha_{15}, 2\alpha_{16}, \alpha_{17}, \alpha_{18}, 2\alpha_{19}, 2\alpha_{20}, 4\alpha_{21}\},$$

$$G_{2,19} = \{4\alpha_1, 4\alpha_2, 3\alpha_4, 4\alpha_5, 4\alpha_6, \alpha_8, 4\alpha_9, 4\alpha_{10}, 2\alpha_{11}, 4\alpha_{12}, 3\alpha_{13}, 4\alpha_{14}, 3\alpha_{15}, \alpha_{16}, 3\alpha_{17}, \alpha_{18}, 2\alpha_{19}, 2\alpha_{20}, 4\alpha_{21}\},$$
\[
3\alpha_{18}, 3\alpha_{19}, \alpha_{20}, 4\alpha_{21},
\]
\[
W_{59,16} = 1 + 48y^{44} + 15y^{48}, \quad W_{59,17} = 1 + 51y^{44} + 9y^{48} + 3y^{52},
\]
\[
W_{59,18} = 1 + 54y^{44} + 3y^{48} + 6y^{52}, \quad W_{59,19} = 1 + 54y^{44} + 6y^{48} + 3y^{56},
\]
(3) If \( s \geq 3 \), then \( N_1(80,3) = 26 \). The 26 non-equivalent optimal \([80,3,60]\) SO codes without zero coordinates have generate matrices \( G_{s,i} (1 \leq j \leq 26) \), and their weight polynomials are \( W_{80,j} \) as follow:
\[
G_{21+17,j} = ((s-3)G_{s}, G_{s,j}, W_{21+17,j} = 1 + y^{16(r-3)}(W_{59,j} - 1), \) when \( (s \geq 3, 1 \leq j \leq 26) \)
\[
G_{s,j} = (G_{s}, G_{s,j})(1 \leq i \leq 19),
\]
\[
G_{s,20} = (4\alpha_1, \ldots, 4\alpha_6, 4\alpha_8, \ldots, 4\alpha_{21}),
\]
\[
G_{s,21} = (5\alpha_1, 5\alpha_2, 5\alpha_3, 5\alpha_4, 5\alpha_5, 4\alpha_6, 3\alpha_7, 3\alpha_8, 4\alpha_9, \ldots, 4\alpha_{15}, 3\alpha_{16}, 4\alpha_{17}, \ldots, 4\alpha_{19}, 3\alpha_{20}, 4\alpha_{21}),
\]
\[
G_{s,22} = (5\alpha_1, 5\alpha_2, 5\alpha_3, 4\alpha_4, 5\alpha_5, 3\alpha_6, 3\alpha_7, 5\alpha_8, 4\alpha_9, 4\alpha_{10}, 3\alpha_{11}, 5\alpha_{12}, 3\alpha_{13}, 4\alpha_{14}, 5\alpha_5, 3\alpha_{16},
\]
\[
5\alpha_{17}, 4\alpha_{18}, 3\alpha_{19}, 3\alpha_{20}, 5\alpha_{21},
\]
\[
G_{s,23} = (5\alpha_1, 5\alpha_2, 2\alpha_3, 4\alpha_4, 4\alpha_5, 5\alpha_6, 2\alpha_8, 5\alpha_9, 4\alpha_{10}, 4\alpha_{11}, 5\alpha_{12}, 4\alpha_{13}, 4\alpha_{14}, 5\alpha_{15}, 2\alpha_{16},
\]
\[
5\alpha_{17}, 4\alpha_{18}, 4\alpha_{19}, 2\alpha_{20}, 5\alpha_{21},
\]
\[
G_{s,24} = (5\alpha_1, 5\alpha_2, \alpha_3, 4\alpha_4, 5\alpha_5, 5\alpha_6, \alpha_8, 5\alpha_9, 5\alpha_{10}, 4\alpha_{11}, 5\alpha_{12}, 4\alpha_{13}, 5\alpha_{14}, 5\alpha_{15}, \alpha_{16},
\]
\[
5\alpha_{17}, 5\alpha_{18}, 4\alpha_{19}, 2\alpha_{20}, 5\alpha_{21},
\]
\[
G_{s,25} = (5\alpha_1, 5\alpha_2, 2\alpha_3, 3\alpha_4, 5\alpha_5, 5\alpha_6, 2\alpha_8, 5\alpha_9, 5\alpha_{10}, 3\alpha_{11}, 5\alpha_{12}, 3\alpha_{13}, 5\alpha_{14}, 5\alpha_{15}, 2\alpha_{16},
\]
\[
5\alpha_{17}, 5\alpha_{18}, 3\alpha_{19}, 2\alpha_{20}, 5\alpha_{21},
\]
\[
G_{s,26} = (5\alpha_1, 5\alpha_2, 5\alpha_3, 5\alpha_4, \ldots, 5\alpha_6, 5\alpha_9, \ldots, 5\alpha_{15}, 5\alpha_{17}, \ldots, 5\alpha_{19}, 5\alpha_{21}),
\]
\[
W_{80,20} = 1 + 48y^{60} + 15y^{64}, \quad W_{80,21} = 1 + 51y^{60} + 9y^{64} + 3y^{68},
\]
\[
W_{80,22} = 1 + 54y^{60} + 3y^{64} + 6y^{68}, \quad W_{80,23} = 1 + 54y^{60} + 6y^{64} + 3y^{72},
\]
\[
W_{80,24} = 1 + 57y^{60} + 3y^{64} + 3y^{76}, \quad W_{80,25} = 1 + 57y^{60} + 3y^{68} + 3y^{72},
\]
\[
W_{80,26} = 1 + 60y^{60} + 3y^{80}.
\]

4. Conclusion

We have given the complete classification of quaternary \([21s+17,3]\) optimal SO codes for \( s \geq 1 \), our results of \([38,3]\) optimal SO codes are concordant with the results of \([7]\). Our classification method given in Section 2 differ from that of \([6]\) and can be generalized to \([n,k]\) optimal SO codes for \( k \geq 3 \).

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6. References