High-Rate Full-Diversity Space-Time-Frequency Code for Multiuser MIMO-OFDM Systems over Frequency Selective Multiple Access Channels

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Abstract

In this paper, we have proposed a new space-time-frequency (STF) code for multiuser multiple-input-multiple-output orthogonal-frequency-division-multiplexing (MIMO-OFDM) systems to increase the spectral efficiency. For each user, the proposed code can achieve high symbol rate (rate-$A_t$) with full diversity ($A_t A_r A_u L$), where $A_t$ denotes the number of transmit antenna for each user, $A_r$ denotes number of receive antenna at the receiver, $A_u$ represents the number of independent fading block in the codeword and $L$ denotes the number of independent channel taps. This STF code has been constructed from a threaded algebraic layering concept, which combines the space-frequency layering with algebraic component codes. The component code is considered to be an algebraic number theoretic constellation. Each component code is assigned to a “thread” and interleaved over space time and frequency. Diophantine approximation theory is then used to make the threads transparent to each other. In addition, another approximation is used so that the users become transparent to each other. Our STF code does not require the cooperation among the users in uplink process and also does not require zero-padding, which always ensures high symbol rate. The proposed coding schemes are bandwidth efficient as the data streams of all the users are sent simultaneously through all the OFDM sub-channels. Simulation results show that our proposed coding schemes achieve higher coding and diversity gain over recently proposed space-frequency codes.

Keywords: MIMO-OFDM Systems, Wideband Multipath Fading Channels, Multiple Access Channel (MAC), Multiuser Space-time-frequency (STF) Coding, and Threaded Algebraic Layering Concept.

1. Introduction

The next generation broadband wireless communication system is required to support robust performance, higher data rate and low-complexity data processing which are capable of providing high-quality multimedia service, to a great number of users anywhere, any time. However, in practice the broadband channel along with the limited spectrum and resource includes much impairment such as frequency selective and time-selective fading which are the major challenges in designing the future wireless communication systems. One promising solution of these issues is the combination of multiple-input-multiple-output (MIMO) with orthogonal frequency division multiplexing (OFDM) such as IEEE 802.11n [1-5]. As a result of this interest, a large number of space-time (ST) and space-frequency (SF) coding have been designed for MIMO-OFDM system [6]-[13] to exploit the spatial diversity gain. However, most of the works focuses mostly on employing single-user space-frequency/time codes and separating the users in signal space or canceling multiuser interference. But the short comings of these approaches are that they sometimes reduce the transmission rate or suboptimum performance significantly, if the number of users is high. A methodical study of the general problem of space time/ frequency code design for MACs seems to be missing.

Gärtner and Bölcskei considered these issues and designed a multi-user space-time/frequency code [14] that was the extension of Gallager’s idea in [15], which is based on the dominant error mechanisms in two-user Additive White Gaussian noise (AWGN) multiple access channel (MAC) but
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didn’t provide a systematic code design. Recently, Zang and Letaief in [1] introduced a systematic design of full diversity multi-user space-frequency code. The symbol or code rate (the definition of code rate is given in [6]) of their proposed design is 1. But to achieve high speed wireless communication high rate data transmission is necessary. However, increasing the code rate is always challenging because it sometimes deteriorate the performance of the code. Though some attentions have been given in high rate full diversity code design, but those were not designed for multi-user MIMO-OFDM system. Therefore, to achieve high symbol rate for multiuser MIMO-OFDM system, in our paper we have proposed a coding schemes for multiuser space-time-frequency (STF) codes, where the proposed scheme can achieve high-rate (rate-$A_t$) and full-diversity for MIMO frequency-selective fading MAC. Here, $A_t$ represents the number of transmitter used in the MIMO system.

To increase the code rate and achieve full-diversity, our multi-user STF codes are constructed by exploiting the space-frequency layering concept with algebraic component code, where the component code is considered to be an algebraic number theoretic constellation such as QAM. Each component code is assigned to a thread in the space-time-frequency matrix that provides full access to the channel frequency, temporal and spatial diversity in absence of other threads. Diophantine approximation theory is then used to make the threads transparent to each other. Further another Diophantine approximation is used to make the users transparent to each other. The numbers are referred to as “Diophantine numbers” because they are chosen in such a way that their simultaneous Diophantine approximation by algebraic number is “bad”, that makes the threads and the users become transparent to each other. Assuming that the channel statistic is known at the transmitter and the instantaneous channel state information (CSI) at the receiver with maximum likelihood (ML) detection using the sphere decoder, our proposed multi-user STF codes for MIMO frequency-selective fading MAC achieve high rate and full-diversity for each user without deteriorating the performance. It is worth noting that the proposed coding scheme does not need the instantaneous channel side information at transmitter nor the cooperation of multiple transmitters.

The organization of the paper is as follows. A system model of the multi-user MIMO-OFDM system has been discussed in section 2. In section 3, the code design criteria for multiple accesses channel (MAC) for multi-user STF codes has been given. The proposed systematic design of multi-user high rate full diversity STF codes has been explained with a proof in section 4. Section 5 has provided an examples to verify our proposed code design. Simulation result and discussion have been given in section 6. Finally, we draw our conclusion in section 7.

Throughout this paper, we will use the following notations. $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ stands for the integer ring, the rational number field, and complex number field respectively. $\mathbb{Q}(j)$ represents the field generated by $j$ and rational where $j=\sqrt{-1}$. $\mathbb{Z}[j]$ denotes the field generated by the $j$ and integer ring. $\lceil x \rceil$ represents the smallest integer larger than $x$, $\lfloor x \rfloor$ represents the largest integer smaller then $x$. The subscripts $T$ and $H$ denotes the transpose and Hermitian of a complex matrix respectively. $\otimes$ and $*$ denotes the kronecker and Hadamard product respectively.

2. System Model

Suppose that in a single-cell network, the MIMO-OFDM system has total number of $Z$ users, where each user is equipped with $A_t$ transmit antennas, a base station (BS) with $A_r$ receive antennas and $N$-tone OFDM. Let, $L$ denotes the number of resolvable channels paths. The MIMO channels experiences wideband multi-path fading induced by $L$ independent paths between each pair of transmit-receive antennas. It is assumed that the MIMO wideband fading channels are subject to block-fading; that means the path gains between each pair of transmit and receive antennas are constant over one fading block, but varies independently in another fading block. Let, there are total number of $A_u$ independent fading blocks in the code word and each OFDM symbol is transmitted during a particular fading block represented by $d$, where $d=1,\ldots, A_u$.

For user $z$ ($z=1,\ldots,Z$), the multipath fading channel from the transmit antenna $n$ to the receive antenna $m$ during the fading block $d$ is assumed to be given by,
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\[ h_{m,n}^{(z)}(t) = \sum_{a=0}^{L-1} h_{m,n}^{(z)}(a) \delta(t - \tau_a), \] (1)

where, \( \tau_a \) is the delay of the \( a^{th} \) path and \( h_{m,n}^{(z)}(a) \) are zero mean, independent identically distributed (i.i.d) complex Gaussian random variables with variance \( \delta_a^2 \) which denotes the channel coefficient of the \( a^{th} \) path from the transmit antenna \( n \) to the receive antenna \( m \). The \( h_{m,n}^{(z)}(a) \) are independent of any \( (m,n,a) \) , where, \( m=1,\ldots,A_r; n=1,\ldots,A_t; \) and \( a = 0, 1, 2, \ldots, L-1 \). The \( \delta(t - \tau_a) \) represents the unit impulse function that determines the specific multi-path bins that have components at time \( t \) and excess delay \( \tau_a \). This implies that the MIMO channels can be assumed to be spatially uncorrelated.

The system has been developed for the coherent scenario where the channel state information (CSI) \( h_{m,n}^{(z)}(a) \) is perfectly known at the receiver but unknown at the transmitter. The statistics of the channel \( h_{m,n}^{(z)}(t) \) are given by \( \delta_a^2 \) and \( \tau_a \), for \( a = 0, 1, 2, \ldots, L-1 \). It is assumed that the channel statistic (\( \tau_a \)) is known at the transmitter.

Furthermore, it is assumed that all path gains between any pair of transmit and receive antennas during any fading block follow the same power profile given by, \( E\left[|h_{m,n}(a)|^2\right] = \delta_a^2 > 0 \). Following the water-filling condition of the optimization problem, the power for \( L \) number of paths are normalized such that \( \sum_{a=0}^{L-1} \delta_a^2 = 1 \).

The above assumption regarding the multipath fading channel model have also been used in other works such as [5],[6],[8],[9],[11-13].

Let,

\[ h_{m,n}^{(z)} = \left[ h_{m,n}^{(z)}(0), \ldots, h_{m,n}^{(z)}(L-1) \right], \] (2)

denotes the complex channel between the \( n^{th} \) transmit antenna of user \( z \) and \( m^{th} \) receive antenna of BS during the fading block \( d \) for, \( m=1,\ldots,A_r; n=1,\ldots,A_t; \) and \( z=1,\ldots,Z; \)

Let, \( H_{m,n}^{(z)} \) denotes the frequency response of the channel from the \( n^{th} \) transmit antenna of user \( z \) to the \( m^{th} \) receive antenna of BS during a single fading block \( d \). It can be given by the Discrete Fourier Transform (DFT) of the channel impulse response as follows,

\[ H_{m,n}^{(z)} = F h_{m,n}^{(z)}, \] (3)

where,

\[ H_{m,n} = \left[ H_{m,n}(0) H_{m,n}(1), \ldots, H_{m,n}(N-1) \right]^T, \] (4)

and,

\[ F = \left[ f^{T_0}, f^{T_1}, \ldots, f^{T_{L-1}} \right]. \] (5)

Here, the column vector \( f \) is given by, \( f = \left[ 1, \zeta, \ldots, \zeta^{N-1} \right]^T \) and \( \zeta = e^{-j2\pi/T_s} \), where \( T_s \) denotes the duration of one OFDM symbol.

The source generates a block of \( N_s \) information symbols from the discrete alphabet \( T \), which are quadrature amplitude modulation (QAM) normalized into the unit power. Using a mapping the information symbol vector \( S \in T^{N_s} \) is encoded into \( N \times A_r A_t \) code matrix \( C^{(z)} \in T^{N \times A_r A_t} \). The symbol rate or code rate per channel use of the code matrix \( C^{(z)} \) is given by, \( R = \frac{N_s}{N} \).

The codeword for user \( z \) can be written as,
where, the \( N \times A_t \) matrix \( C^{d(z)} \) (where \( d = 1, \ldots, A_u \)) denotes the sub-codeword that is considered to be sent during the fading block \( d \).

Let, each user has \( N \)-tone OFDM. For user \( z \), the codeword \( C^{d(z)} \) can be written as,

\[
C^{d(z)} = \begin{bmatrix}
X_1^{d(z)} \\
\vdots \\
X_{A_t}^{d(z)}
\end{bmatrix}
\]

(7)

where the OFDM symbol \( X_n^{d(z)} \) is assumed to be transmitted from the \( n \)th transmit antenna.

Let, two codeword \( C^{d(z)} \) and \( C^{d(z')}, \) of two different users \( z \) and \( z' \) are independent. Assuming the perfect synchronization is available at the receiver, the received signal at the BS can be given by,

\[
Y_m^{d(z)} = \sum_{n=1}^{A_t} \sum_{z=1}^{L-1} E_n X_n^{d(z)} h_{m,n}^{d(z)} (a) + n_m^{d(z)},
\]

(8)

where,

\[
Y_m^{d(z)} = \left[ Y_m^{d(z)}(0) Y_m^{d(z)}(1) \ldots Y_m^{d(z)}(N-1) \right]^T,
\]

(9)

and \( n_m^{d(z)} \in \mathbb{C}^N \) is Additive White Gaussian Noise (AWGN) with zero mean and covariance \( N_0 I_N \). \( Y_m^{d(z)} \in \mathbb{C}^N \) is the received signal at the \( m \)th antenna of base station, \( E_n = diag(f^{za}) \) and \( E_n X_n^{d(z)} = diag \left( X_n^{d(z)} \right) f^{za} \).

Moreover, compacting some terms in (8) and using the following notations,

\[
p^{d(z)} = \left[ E_0 C^{d(z)} \ldots E_{L-1} C^{d(z)} \right]
\]

(10)

\[
p^d = \begin{bmatrix} p^{d(1)} & \ldots & p^{d(Z)} \end{bmatrix}
\]

(11)

\[
X^d = I_{A_t} \otimes p^d,
\]

(12)

we get,

\[
Y^d = \sqrt{\frac{\rho}{A_t}} X^d h^d + n^d,
\]

(13)

The term \( \sqrt{\frac{\rho}{A_t}} \) ensures that the SNR at each receive antenna \( \rho \), independently of the transmitter \( A_t \).

3. STF Code Design Criterion

The code design criterion for space-time-frequency code is based on the error event analysis which was first discussed by Gallager [15]. Following this, Gartner and Bolcskei [14] has established a ST code design criteria for multiple access channel (MAC), but it was only for two users. Addressing this issue, a systematic design of full diversity SF codes for any number of users has been established [1]. Depending on the desired transmission rate tuple, the dominant error event regions for the two users MAC was discussed [15]. It was shown that for the rate region with user-1 or user-2 is in error, the well known single user ST/SF codes are sufficient. But, for the rate region where both users are in error, a joint code design is necessary for optimization, i.e.: to get full-diversity for any error event. Moreover, the code design criteria were made based on high-SNR and low-SNR region. As, this effect is less pronounced for low SNR, our code design criteria for STF code is based on the high-SNR region.

During the fading block \( d \), for a single user (i.e. \( z=1 \)) codeword \( C^{d(z)} \), the pair wise error probability (PEP) can be given by \( P(C^{d(z)} \rightarrow C^{d(z')}) \), where \( C^{d(z)} \neq C^{d(z')} \). Denoting the number of OFDM tones
as $N$, for a given channel realization $H_d\left( e^{j2\pi k/N} \right)$ on the $k$-th tone (for, $k = 0, \cdots , N-1$), the probability that the receiver decides erroneously in favor of $\hat{C}^{d(z)}$ is given by as [16],

$$P \left( C^{d(z)} \to \hat{C}^{d(z)} \mid H_d \left( e^{j2\pi k/N} \right) \right) = \frac{E_s}{2\sigma^2 d^2} \left( \frac{C^{d(z)}, \hat{C}^{d(z)} \mid H_d \left( e^{j2\pi k/N} \right)}{2\sigma^2} \right),$$

where, the squared Euclidian distance between the codeword $C^{d(z)}$ and $\hat{C}^{d(z)}$ denoted by,

$$d^2 \left( C^{d(z)}, \hat{C}^{d(z)} \right) H_d \left( e^{j2\pi k/N} \right) = \left\| \left( I_{A_d} \otimes \hat{C}^{d(z)} \right) H_d \left( e^{j2\pi k/N} \right) \right\|^2.$$

Denoting SNR as $\rho = \frac{E_s}{\sigma^2 n}$ and taking $H_d = H_d \left( e^{j2\pi k/N} \right)$, $C^{d(z)} = C^{d(z)} - \hat{C}^{d(z)}$; using Chernoff bound $Q(x) \leq e^{-x^2}$, the PEP can be upper bounded as,

$$P \left( C^{d(z)} \to \hat{C}^{d(z)} \mid H_d \right) \leq e^{-\frac{\rho}{4\sigma^2 d^2} ||(I_{A_d} \otimes C^{d(z)})H_d||^2}.$$  

Since, every Eigen value of the $A_d \times A_d$ matrix $\left( C^{d(z)} - \hat{C}^{d(z)} \right) \left( C^{d(z)} - \hat{C}^{d(z)} \right)^T$ is an Eigen value of $A_d \times A_d \times A_d$ matrix $\left( C^{d(z)} - \hat{C}^{d(z)} \right) \left( C^{d(z)} - \hat{C}^{d(z)} \right)^T \times I_{A_d}$, (with multiplicity $A_d$).

Let,

$$R^{d(z)} = I_{A_d} \otimes (\hat{C}^{d(z)}H \hat{C}^{d(z)}).$$

Then using singular value decomposition, we get,

$$R^{d(z)} = U^H \Lambda U,$$

where, $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_{W_LW_TW_\gamma A_r})$, $W_L = 2^{[\log_2 L]}$, $W_\gamma = 2^{[\log_2 A_\gamma]}$ with, $\lambda_f \geq 0$ for $f = 1, \cdots, W_LW_TW_\gamma A_r$.

Therefore,

$$\left\| \left( I_{A_d} \otimes \hat{C}^{d(z)} \right) H_d \right\|^2 = \left\| \left( I_{A_d} \otimes \hat{C}^{d(z)} \right) H_d \right\|^2 \left\| U^H \Lambda U \right\|^2.$$  

Let, $\gamma = UH_d$ and $\gamma_f$ denotes the $f$-th element of the vector $\gamma$. Then from (19), we get,

$$\left\| \left( I_{A_d} \otimes \hat{C}^{d(z)} \right) H_d \right\|^2 = \sum_{f=1}^{W_LW_TW_\gamma A_r} \lambda_f |\gamma_f|^2.$$  

Substituting (19) into (16) we get,

$$P \left( C^{d(z)} \to \hat{C}^{d(z)} \right) \leq e^{-\left( \frac{\rho}{4\sigma^2 d^2} \sum_{f=1}^{W_LW_TW_\gamma A_r} \lambda_f |\gamma_f|^2 \right)},$$

Taking the expectation of (21) we get,

$$P \left( C^{d(z)} \to \hat{C}^{d(z)} \right) \leq \prod_{f=1}^{r} \left( 1 + \rho \frac{\lambda_f}{\gamma_f} \right),$$

where, $r$ is the rank of the matrix $R^{d(z)}$. As our STF code design criteria is based on high-SNR thus for high-SNR, $\rho \gg 1$, simplifying (22),

we get,

$$P \left( C^{d(z)} \to \hat{C}^{d(z)} \right) \leq \rho^{-r} \prod_{f=1}^{r} \frac{1}{\lambda_f}.$$  

Now,

$$R^{(z)} = \text{diag} \left( R_1^{(z)} \cdots R_{A_d}^{(z)} \right).$$
Let, $\lambda_1, \lambda_2, ..., \lambda_{n_w}$ and $\lambda_{d, 1}, \lambda_{d, 2}, ..., \lambda_{d, n_w}$ are the eigen values of $R^{d(z)}$ and $R(z)$, where, $n_w$ denotes the diversity order of the error event when $w$ out of $Z$ users are in error.

For 2-user system (i.e. $z$=1,2), let, during the fading block $d$, the transmitted codeword is denoted by, $C^d = \{ C^{d(1)} C^{d(2)} \}$. Let, $\hat{C}^d$ denotes the detected codeword. Assume three types of error event as type-1, type-2, type-3, where type-1 and type-2 represents the error events when a user-1 or user-2 is in error, and type-3 represents the error events when both users are in error. Thus for a given channel realization, $R^d$, the total average pair wise error probability is given by[14],

$$P_e = P_{eq1} + P_{eq2} + P_{eq3}. \tag{25}$$

where, $P_{eq} = E\left[p_{eq}\mid I^d\right]$ and $P_{eq}\mid I^d$ for $q=1, 2, 3$.

The three terms in (25) depicts all error events when the pair wise error $C^d \neq \hat{C}^d$ occur.

For a general case of total number of $Z$ users, let, the code word $C^d = \{ C^{d(1)} C^{d(2)} ... C^{d(z)} \}$ and $\hat{C}^d$ denotes the detected code word. Assume, $\hat{C}^d = C^d - \hat{C}^d$ and $\bar{C}^{d(z)} = C^{d(z)} - \hat{C}^{d(z)}$. then the probability of symbol error can be upper bounded as [1],

$$P_e \leq \sum_{|D|=1} \left( \prod_{j=1}^{z} \left( \frac{1}{\lambda_j} \right) \left( P(|D| = 1) \right) \right) \sum_{|D|=z} \left( \prod_{j=1}^{z} \left( \frac{1}{\lambda_j} \right) \left( P(|D| = z) \right) \right), \tag{26}$$

where, $D=\{z|C^{d(z)} \neq \bar{C}^{d(z)}\}$ where $|D| \geq 1$ for $C^{d(z)} \neq \bar{C}^{d(z)}$ and $r_w$ is the rank of the matrix $R^d = L_r \otimes (\hat{C}^H \bar{C}^d)$, where only $w$ out of $Z$ user have $\bar{C}^{d(z)} \neq 0$. The probability that only $w$ out of $Z$ users have $C^{d(z)} \neq \bar{C}^{d(z)}$ is denoted by $P(|D| = w)$.

Now, $P_{eq}$ is dominated by the codeword difference matrices $R^{d(z)}$ with minimum rank [14]. As $n_w$ denotes the diversity order of the error event when $w$ out of total number of $Z$ users are in error, the full-diversity for every error event should be guaranteed by the code design. Thus in this coherent scenario, the code design criteria of full diversity SF codes for MAC over MIMO frequency-selective block-fading channels as follows:-

a. **Rank criterion:**
Maximize the transmit diversity gain over all pairs of distinct codeword $C(x)$ and $\hat{C}(x)$,

$$r = \sum_{w=1}^{A_u} \sum_{d=1}^{A_u} A_{r_w} \cdot rank(C^{d(x)} - \hat{C}^{d(x)}), \tag{27}$$

when only $w$ out of $Z$ users have $C^{d(z)} \neq \hat{C}^{d(z)}$ for $w=1... Z$.

b. **Block fading product criterion:**
Maximize the coding gain over all pairs of distinct code words $C(x)$ and $\hat{C}(x)$,

$$C_w = \prod_{f=1}^{r_w} \lambda_f = \prod_{d=1}^{A_u} \lambda_{d, 1}, \lambda_{d, 2}, ..., \lambda_{d, n_w}, \tag{28}$$

when, only $w$ out of $Z$ users have $C^{d(z)} \neq \hat{C}^{d(z)}$ for $w=1... Z$.

4. **Multiuser Space-Time-Frequency Code Design**

In this section, we have proposed a design of STF code that is capable of achieving full diversity $A_1 A_2 A_3 L$ and high code rate (i.e. rate-$A_1$) over MIMO frequency selective block fading channel. The construction of the proposed multiuser STF is illustrated in fig-1.
Let,
\[
Q = W_x W_L W_t,
\]
where, \(W_L = 2^{[\log_2 L]}\), \(W_t = 2^{[\log_2 A_t]}\), \(W_z = 2^{[\log_2 Z]}\).

The block of \(NA_tA_u\) information symbols, \(S^{(z)} = \left[ S_1^{(z)} \ S_2^{(z)} \ \ldots \ S_{BQ}^{(z)} \right]^T\),

are normalized into the unit power and are evenly split into \(B = \frac{N}{Q}\) sub blocks,
\[
S^{(z)} = \left[ \left(S_1^{(z)}\right)^T \left(S_2^{(z)}\right)^T \ \ldots \ \left(S_{B}^{(z)}\right)^T \right]^T.
\]

Each sub block \(S_b^{(z)} \in \mathcal{T}^{W_xW_LW_tA_tA_u}\), \(b = 1, 2, \ldots, \), \(B\) is composed of the signal vectors \(S_w^{(z)} \in \mathcal{T}^\mathcal{X}\), \(w=1,2,3,\ldots, W_t\) and \(\tilde{w} = W_x W_L A_t A_u\) and given by,
\[
S_{b}^{(z)} = \left[ \left(S_1^{(z)}\right)^T \left(S_2^{(z)}\right)^T \ \ldots \ \left(S_{W_t}^{(z)}\right)^T \right]^T.
\]

Each one of the component vector \(S_w\) is then encoded independently using constituent encoder \(\tilde{x}_w^{(z)} : \mathcal{T}^\mathcal{X} \rightarrow \mathcal{X}^\mathcal{X}\), where, \(\mathcal{X}\) is the output alphabet and given by,
\[
\tilde{x}_w^{(z)} = \Theta S_w^{(z)}
\]
\[
= \left[ x_{w,1}^{A_t} \ x_{w,W_L}^{A_t} \ \ldots \ x_{w,1}^{A_u} \ x_{w,W_L}^{A_u} \right]^T,
\]

where, \(\Theta\) is a \(\mathcal{W} \times \mathcal{W}\) unitary matrix (rotational matrix) and it is constructed by the first principal \(\mathcal{W} \times \mathcal{W}\) matrix of the following \(\tilde{\mathcal{Y}} \times \tilde{\mathcal{Y}}\) matrix,
\[
\psi = F^{\tilde{\mathcal{Y}}} \ diag(1, \varphi, \ldots, \varphi^{\tilde{\mathcal{Y}}}),
\]

where, \(\tilde{\mathcal{Y}} = 2^{[\log_2 W]}\), \(F_{\tilde{\mathcal{Y}}}\) is the \(\tilde{\mathcal{Y}} \times \tilde{\mathcal{Y}}\) discrete Fourier transform (DFT) matrix and \(\varphi = e^{-j\frac{\pi}{2\tilde{\mathcal{Y}}} \mathcal{Y}}\).

It is worth noting that, multiplying information symbol vector \(S_w^{(z)}\) by the rotational matrix \(\Theta\) maximize the associated minimum product distance [17],
\[
d_{w} = \min_{\tilde{x}_w^{(z)} \in \mathcal{X}^\mathcal{X}} \prod_{w=1}^{W_t} \left| \tilde{x}_w^{(z)} \right|,
\]

---

**Fig 1.** STF coding structure in MIMO-OFDM system
where, $X^{(z)} = \left[ X_1^{(z)} \ X_2^{(z)} \ \ldots \ X_{w_t}^{(z)} \right]^T$.

As coding gains are proportional to the minimum product distances associated with the rotational matrix used, the maximization of the coding gain in the code design criteria is achieved.

Each one of the encoded component vector $S_w^{(z)}$ is then multiplied by the Diophantine number $\phi_{1,w}$ to ensure full diversity and maximize the coding gain for the joint code. The numbers are referred to as “Diophantine numbers” because they are chosen in a way that their simultaneous Diophantine approximation by algebraic number is “bad”, that makes the threads and the users become transparent to each other. Each $\phi_{1,w}$ is chosen from the $w^\text{th}$ diagonal layer of the $W_t \times A_t$ matrix,

$$
\phi_1 = \begin{pmatrix}
1 & \phi_1^{(w_t-1)} & \ldots & \phi_1^{(w_t-A_t)+1} \\
\phi_1 & 1 & \ldots & \phi_1^{(w_t-A_t)+2} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_1^{(w_t-1)} & \phi_1^{(w_t-2)} & \ldots & \phi_1^{(1-A_t)w_t}
\end{pmatrix}
$$

(37)

$\phi_1$ will be defined latter. The term $\hat{x}_{d,i}^{(z)}$ in (34) is given by,

$$
\hat{x}_{d,i}^{(z)} = \left[ X_w^{(z)}(P_{i,d}^l + 1) \ldots X_w^{(z)}(P_{i,d}^l + A_t) \right],
$$

(38)

for $i=1, \ldots, W_L$, where the index,

$$
P_{i,d}^l = (i-1)A_t + (l-1)W_L A_t + (d-1)W_L W_L A_t,
$$

(39)

where $l=1,\ldots, W_L$.

Next, a space-frequency formatter assigns the $w^\text{th}$ code symbols $X_w^{(z)}(P_{i,d}^l + n)$ (where, $n = 1, 2, \ldots, A_t$) of the row vector $\hat{x}_{d,i}^{(z)}$ on the $w^\text{th}$ layer of the $W_t \times A_t$ matrix $\hat{X}_{d,i}^{(z)}$. The construction of the matrix is given by,

$$
\hat{X}_{d,i}^{(z)} = X_{l,i}^{(z)} \circ \phi_1,
$$

(40)

The $W_t \times A_t$ matrix $X_{l,i}^{(z)}$ is given by,

$$
X_{l,i}^{(z)} = 
\begin{pmatrix}
X_1^{(z)}(P_{i,d}^l + 1) & X_{w_1}^{(z)}(P_{i,d}^l + 1) & \ldots & X_{(w_t-A_t)+2}^{(z)}(P_{i,d}^l + 1) \\
X_1^{(z)}(P_{i,d}^l + 2) & X_{w_1}^{(z)}(P_{i,d}^l + 2) & \ldots & X_{(w_t-A_t)+3}^{(z)}(P_{i,d}^l + 2) \\
\vdots & \vdots & \ddots & \vdots \\
X_{w_t}^{(z)}(P_{i,d}^l + A_t) & X_{w_1}^{(z)}(P_{i,d}^l + A_t) & \ldots & X_{(1-A_t)w_t+1}^{(z)}(P_{i,d}^l + A_t)
\end{pmatrix}
$$

(41)

The diagonal layer index of $\hat{X}_{d,i}^{(z)}$ is shown in fig-2 for $A_t=3$ and $W_t=4$.

$$
\begin{pmatrix}
1 & 4 & 3 \\
2 & 1 & 4 \\
3 & 2 & 1 \\
4 & 3 & 2
\end{pmatrix}
$$

Fig 2. example of symbols placement in the diagonal layers of $\hat{X}_{l,i}^{(z)}$ matrix for $A_t=3$ and $w_t=4$.

Thus each sub-block $S_b^{(z)}$, $b=1,\ldots, B$ is encoded into an STF code matrix $c_b^{(z)}$ of size $Q \times A_t A_u$ matrix, where,
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\[ C_b^{(z)} = \begin{bmatrix}
X_{1,1}^{(z)} & \cdots & X_{1,n}^{(z)} \\
X_{1,w_1}^{(z)} & \cdots & X_{1,w_n}^{(z)} \\
\vdots & \ddots & \vdots \\
X_{w_1,1}^{(z)} & \cdots & X_{w_1,n}^{(z)} \\
X_{w,w_1}^{(z)} & \cdots & X_{w,w_n}^{(z)} \\
\vdots & \ddots & \vdots \\
X_{w_1,w_1}^{(z)} & \cdots & X_{w_1,w_n}^{(z)} \\
\vdots & \ddots & \vdots \\
X_{w,w_1}^{(z)} & \cdots & X_{w_1,w_n}^{(z)} \\
\end{bmatrix}. \]

(42)

For multiuser MIMO-OFDM MAC, the encoded codeword \( \tilde{C}_b^{(z)} \) is given by,

\[ \tilde{C}_b^{(z)} = \left( \Phi_{2,z} \otimes \mathbf{1}_{1 \times A_{u}} \right) \circ C_b^{(z)}, \]

where, \( \Phi_{2,z} \) is the \( z^{th} \) column of the \( Q \times W_z \) matrix \( \Phi_2 \),

\[ \Phi_2 = \begin{pmatrix}
1 & \phi_2^{(Q-1)} & \cdots & \phi_2^{(Q-W_1+1)} \\
\phi_2 & 1 & \cdots & \phi_2^{(Q-W_1+2)} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_2^{(Q-1)} & \phi_2^{(Q-2)} & \cdots & \phi_2^{(Q-W_1)Q} \\
\end{pmatrix}. \]

(44)

The choice of \( \phi_1 \) and \( \phi_2 \) are as follows:

- \( \phi_1 = \theta^{w_t} \), where \( \theta \) is an algebraic element with degree at least \( W_t \) over \( \mathcal{A} \)
- \( \phi_2 = \phi^{\frac{1}{w_t}} \), where \( \phi \) is an algebraic number with degree of at least \( QW_t \) over \( \mathcal{A} \), where \( \mathcal{A} \) is the field extension of \( \mathbb{Q} \) which contains the signal alphabet \( T \subset \mathbb{Z}[j, e^{j2\pi k/L}], (l = 0,1, ..., L-1) \) and all the entries of \( \theta \).

It is worth noting that, according to the \textit{Theorem} 1 and \textit{Theorem} 2 given in [17], the coding gain expresses the simultaneous Diophantine approximation of the numbers \( \left\{ \phi_1^0 = 1, \phi_1^1 = \theta^{w_t}, ..., \phi_1^{(w_t-1)} = \theta^{w_t-1} \right\} \) and \( \left\{ \phi_2^0 = 1, \phi_2^1 = \phi^{\frac{1}{w_t}}, ..., \phi_2^{(Q-1)} = \phi^{\frac{Q-1}{w_t}} \right\} \) by other algebraic numbers, depending on the constellation used. This observation implies that optimizing the coding gain is equivalent to choosing these Diophantine numbers to be “badly approximated” by other algebraic number.

The proposed STF coding applies the same coding strategy to every encoded sub-block \( \tilde{C}_b^{(z)} \), \( b = 1, 2, ..., B \). So, for convenience, we have just showed the STF coding of one sub block \( \tilde{C}_b^{(z)} \).

Thus, the proposed STF codes \( C^{(z)} \in \mathbb{T}^{W \times A_t A_t} \) for the \( z^{th} \) user is of the form,

\[ C^{(z)} = \left( \begin{array}{c}
\tilde{C}_1^{(z)} \\
\tilde{C}_2^{(z)} \\
\vdots \\
\tilde{C}_B^{(z)}
\end{array} \right)^T. \]

(45)

Note that, the difference of the SF code between any two users lies in the design of \( \Phi_{2,z} \) in (44) which is selected from the different columns of matrix \( \Phi_2 \). Since \( \Phi_2 \) is fixed in advance, each transmitter knows its \( \Phi_{2,z} \) before transmitting the data and \( \Phi_{2,z} \) is fixed for each user, therefore the cooperation between the users is not necessary in the uplink process.
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It is also worth noting that, the proposed SF code achieve high symbol rate, i.e rate- \( A_t \) per channel use. This can be seen from (45) that \( NA_u \) encoded independent information symbol are sent over \( NA_u \) OFDM tones used in the system. Furthermore, as TASTF code confirms that \( N \) is the integer multiple of \( Q \), no zero-padding matrix is required in our proposed code structure. Thus the high rate (rate- \( A_t \) ) is always guaranteed.

Now, we give the following theorem:

**Theorem:** Suppose that in a single cell network, a multi-user MIMO-OFDM system with \( Z \) user, each having \( A_t \) transmit antennas, one BS having \( A_r \) receiver antennas, has \( N \) OFDM tones; such that the MIMO channels that are spatially uncorrelated experience the wideband block-fading characterized by \( L \) independent paths. Then the proposed multi-user STF codes designed in (29)-(45) achieve full diversity - \( A_t A_r A_u L \) and the symbol rate- \( A_t \) per channel use.

**Proof:** The proof of high rate and full-diversity of proposed code are given in the Appendix.

The maximum likelihood (ML) decoding is often used for full-diversity performance of STF codes. However, the decoding complexity is very large with the increase of the value of \( L \) and \( A_t \) [5]. To remove the burden of ML, sphere decoder can be used to achieve the approximate performance of ML [17], [18] and [20].

5. STF Code Design Example

In this section, we have just shown the sub-block \( \vec{C}_1^{(z)} \) as an example for convenience because the same coding strategies can be applied to every sub-block \( \vec{C}_b^{(z)} \), \( b=1,\ldots, B \).

**Example-:** Let, \( A_u = 2, A_t = 2, L = 2, Z = 2 \). Thus, we get,

\[
\vec{C}_1^{(1)} = \begin{pmatrix}
X_1^{(1)(1)} & \phi_1 X_2^{(1)(1)} & X_1^{(1)(9)} & \phi_1 X_2^{(1)(9)} \\
\phi_2 X_1^{(1)(2)} & \phi_2 X_2^{(1)(2)} & \phi_2 X_1^{(1)(10)} & \phi_2 X_2^{(1)(10)} \\
\phi_3 X_1^{(1)(5)} & \phi_3 X_2^{(1)(5)} & \phi_3 X_1^{(1)(13)} & \phi_3 X_2^{(1)(13)} \\
\phi_4 X_1^{(1)(6)} & \phi_4 X_2^{(1)(6)} & \phi_4 X_1^{(1)(14)} & \phi_4 X_2^{(1)(14)} \\
\phi_5 X_1^{(1)(3)} & \phi_5 X_2^{(1)(3)} & \phi_5 X_1^{(1)(11)} & \phi_5 X_2^{(1)(11)} \\
\phi_6 X_1^{(1)(4)} & \phi_6 X_2^{(1)(4)} & \phi_6 X_1^{(1)(12)} & \phi_6 X_2^{(1)(12)} \\
\phi_7 X_1^{(1)(7)} & \phi_7 X_2^{(1)(7)} & \phi_7 X_1^{(1)(15)} & \phi_7 X_2^{(1)(15)} \\
\phi_8 X_1^{(1)(8)} & \phi_8 X_2^{(1)(8)} & \phi_8 X_1^{(1)(16)} & \phi_8 X_2^{(1)(16)}
\end{pmatrix}
\]

and

\[
\vec{C}_1^{(2)} = \begin{pmatrix}
\phi_2 X_1^{(2)(2)} & \phi_2 X_2^{(2)(2)} & \phi_2 X_1^{(2)(9)} & \phi_2 X_2^{(2)(9)} \\
\phi_3 X_1^{(2)(2)} & \phi_3 X_2^{(2)(10)} & \phi_3 X_1^{(2)(10)} & \phi_3 X_2^{(2)(10)} \\
\phi_4 X_1^{(2)(5)} & \phi_4 X_2^{(2)(5)} & \phi_4 X_1^{(2)(13)} & \phi_4 X_2^{(2)(13)} \\
\phi_5 X_1^{(2)(6)} & \phi_5 X_2^{(2)(6)} & \phi_5 X_1^{(2)(14)} & \phi_5 X_2^{(2)(14)} \\
\phi_6 X_1^{(2)(3)} & \phi_6 X_2^{(2)(3)} & \phi_6 X_1^{(2)(11)} & \phi_6 X_2^{(2)(11)} \\
\phi_7 X_1^{(2)(4)} & \phi_7 X_2^{(2)(4)} & \phi_7 X_1^{(2)(12)} & \phi_7 X_2^{(2)(12)} \\
\phi_8 X_1^{(2)(7)} & \phi_8 X_2^{(2)(7)} & \phi_8 X_1^{(2)(15)} & \phi_8 X_2^{(2)(15)} \\
\phi_9 X_1^{(2)(8)} & \phi_9 X_2^{(2)(8)} & \phi_9 X_1^{(2)(16)} & \phi_9 X_2^{(2)(16)}
\end{pmatrix}
\]

where, the sub-block \( \vec{C}_1^{(1)} \) is for user, \( z=1 \) and the sub-block \( \vec{C}_1^{(2)} \) is for user, \( z=2 \). Note that, when \( A_u=1 \), the STF code has only left half portion of above example which coincides with the design.
of rate- $A_u$ full-diversity multiuser SF code proposed in [19]. Therefore, for $A_u=1$, the STF code can be viewed as the SF code.

6. Simulation Results:

In our simulation, two users scenario has been considered where each user has two transmit antennas and the base station has two receive antennas with equal power gain. A two ray channel model has been simulated for each pair of transmit-receive antennas. The length of the cyclic prefix is chosen as 16 and $N=64$ OFDM tones is used for each transmit antenna. The second path delay is assumed to be $0.5\mu s$ that is, 10 times the sampling interval. In the simulation, the channel coefficients are independent from one OFDM block to other block but are remain constant during one OFDM block. Regarding the proposed STF code, the total number of $A_u=2$ independent fading block in the code word has been considered for the simulation. The simulation result shows the performance comparison among the multiuser SF code [1], Alamouti code [21] and the proposed multiuser space-time-frequency (STF) code, with 16-QAM.

Fig-3 shows that the symbol error probability (SER) $P_{e,3}$ for both the single user ST code [21] (Alamouti code) and the proposed multiuser STF code with 16-QAM. It is apparent that the Alamouti code can achieve full special diversity gain of 4; on the other hand the proposed multiuser STF code achieve full spatial, temporal and frequency diversity gain of $A_t A_r A_u L = 16$. From the graph, it can be seen that the proposed code have larger slope diversity gain than the Alamouti code.

Fig 3. SER performance comparison among the multiuser SF code [1], Alamouti code [21] and the proposed multiuser space-time-frequency code.

Another SER performance comparison between the proposed code and the multiuser SF code in [1] has been given in fig-3. From the analysis it can be seen that the SF code in [1] achieve full spatial and frequency diversity gain of $A_t A_r L = 8$, whereas the proposed code can achieve full spatial, temporal and frequency diversity of $A_t A_r A_u L = 16$. From the graph, it can be seen that the proposed STF code has larger slope curve compared with [1]. This indicates that the proposed STF codes achieve a higher
diversity gain than the SF code [1]. Moreover, it is observed that the proposed rate-$A_t$(in this case $A_t=2$) STF code achieve the best SER performance among the simulated cases and outperforms with a gain of about 2 dB over all examined SNR region when it is compared with Alamouti code[21] and SF code [1]. The analysis implies that the proposed multiuser STF code has a better coding gain than the both rate-1 SF code[1] and Alamouti code[21]. The diversity and coding gain advantages are attributed to the proposed multiuser STF codes which can obtain full diversity and high rate (rate - $A_t$) for every error event of the two user system which indicates that the increment of code rate does not deteriorate the performance.

7. Conclusion

In this paper, we have proposed a new design of space-time-frequency codes for MIMO-OFDM systems. The high-rate (rate- $A_t$) and full-diversity $A_tA_tA_tL$ has been achieved which validate the theoretical analysis. The proposed multi-user STF code is bandwidth efficient and always ensures high rate- $A_t$. An example of our proposed code design has been given for better understanding of our proposed code. Simulation result proves that this STF code has achieved higher diversity and coding gain than the existing code for all SNR regions.

APPENDIX

A. Proof of high symbol rate-$A_t$:
The symbol rate of the proposed multiuser STF codes can be calculated by the formula,

$$\mathcal{R} = \frac{NA_tA_u}{NA_u} = A_t.$$  

B. Proof of full-diversity—$A_tA_tA_tL$:

Assume that $S^{(z)} - \hat{S}^{(z)} \neq 0$ for all $z=1,...,Z$. From (31) it is seen that there exists at least one index $b_z \in [1, B]$ for user $z$ such that, $S_{b_z}^{(z)} \neq \hat{S}_{b_z}^{(z)}$ for all $z=1,...,Z$. So, from (34), the matrix $\hat{X}_w = X_w - \hat{X}_w$ has no zero entries for all $z$.

For any point of distinct SF code word $C^{(z)}$ and $\hat{C}^{(z)}$, from (45) there exists at least one index $b \in [1, B]$ such that $C_b^{(z)} \neq \hat{C}_b^{(z)}$ and $S_b^{(z)} \neq \hat{S}_b^{(z)}$.

Using (42), we can express $C^{(z)} \in T^{Q \times A_tA_u}$ as,

$$C^{(z)}_b = [C^{(z)}_b \ldots C^{(z)}_{A_tA_u}],$$

where,

$$C^{(z)}_b = [C^{(z)}_{b,1} \ldots C^{(z)}_{b,W_{L}}],$$

and

$$C^{(z)}_{b,i} = \left[\begin{array}{c}X_{L,1}^{d(z)} \ldots (X_{L,2}^{d(z)})^T \ldots (X_{L,W_{L}}^{d(z)})^T \end{array}\right]^T,$$

for $l=1,...,W_{L}$ and $i=1,...,W_{L}$.

Let, $\tilde{C}^{(z)} = C^{(z)} - \hat{C}^{(z)}$. Using (10), the $N \times A_tL$ matrix $\tilde{G}^{(z)}$ can be given by,

$$\tilde{G}^{(z)} = \left[E_0 \tilde{C}^{(z)} E_1 \tilde{C}^{(z)} \ldots E_{L-1} \tilde{C}^{(z)}\right]Q^T.$$

Next, we assume,

$$\tilde{C}^{d} = \left[G^{(1)} \ldots G^{(Z)}\right].$$
It can be seen that $C^{d(z)}_p \in T^{Q \times A_1}$ is the sub matrix of $C^{d(z)} \in T^{N \times A_1}$ and composed of Q rows of $C^{d(z)}$. Thus the $Q \times A_1L$ matrix $T^d$ can be composed by taking Q rows of $G^{(z)}$ as,

$$
T^d = \begin{pmatrix}
\delta_0 E_{0,h,1} \tilde{X}^{d(z)}_{1,1} & \cdots & \delta_{L-1} E_{L-1,h,1} \tilde{X}^{d(z)}_{1,1} \\
\vdots & \ddots & \vdots \\
\delta_0 E_{0,h,W_L} \tilde{X}^{d(z)}_{1,W_L} & \cdots & \delta_{L-1} E_{L-1,h,W_L} \tilde{X}^{d(z)}_{1,W_L} \\
\delta_0 E_{0,h,1} \tilde{X}^{d(z)}_{W_L+1,1} & \cdots & \delta_{L-1} E_{L-1,h,1} \tilde{X}^{d(z)}_{W_L+1,1} \\
\vdots & \ddots & \vdots \\
\delta_0 E_{0,h,W_L} \tilde{X}^{d(z)}_{W_L+1,W_L} & \cdots & \delta_{L-1} E_{L-1,h,W_L} \tilde{X}^{d(z)}_{W_L+1,W_L}
\end{pmatrix}
$$

(A6)

where, $\tilde{X}^{d(z)}_{i,l} = X^{d(z)}_{i,l} - X^{d(z)}$ for $i=1,...,W_L$, $l=1,...,W_L$ and $E_{l,b,i}$ is the $W_t \times W_t$ matrix for $z=1,...,Z$. The $w^{th}$ element on the diagonal of the matrix is given by $|5|,$

$$
E_{a,b,i}^{(w)} = (\zeta^{(b-1)q+(i-1)Lw+w-1})^{T_1},
$$

where, $a=0,1,...,L-1$, $i=1,...,W_L$, and, $\zeta = \exp\left(\frac{-j\pi}{T_2}\right)$.

Next, the $W_t \times W_t$ matrix $\tilde{X}^{d(z)}_{i,l}$ is designed by adding $W_t-A_t$ column to the $W_t \times A_t$ matrix $\tilde{X}^{d(z)}_{i,l}$ and $i=1,...,W_L$, $l=1,...,W_L$.

Replacing, $\tilde{X}^{d(z)}_{i,l}$ in (A6) by $\tilde{X}^{d(z)}_{i,l}$ and deleting bottom $Q - W_LW_tL$ rows of (A6) we have obtained the following $W_tW_tL \times W_tL$ matrix,

$$
\tilde{T}^d = \begin{pmatrix}
\tilde{E}_{0,b,1,1} \tilde{X}^{d(z)}_{1,1} & \cdots & \tilde{E}_{L-b,1,1} \tilde{X}^{d(z)}_{1,1} \\
\vdots & \ddots & \vdots \\
\tilde{E}_{0,b,1,W_L} \tilde{X}^{d(z)}_{1,W_L} & \cdots & \tilde{E}_{L-b,1,W_L} \tilde{X}^{d(z)}_{1,W_L} \\
\tilde{E}_{0,b,1} \tilde{X}^{d(z)}_{W_L+1,1} & \cdots & \tilde{E}_{L-b,1} \tilde{X}^{d(z)}_{W_L+1,1} \\
\vdots & \ddots & \vdots \\
\tilde{E}_{0,b,1} \tilde{X}^{d(z)}_{W_L+1,W_L} & \cdots & \tilde{E}_{L-b,1} \tilde{X}^{d(z)}_{W_L+1,W_L}
\end{pmatrix}
$$

(A7)

where, the terms $\delta_a \neq 0$ for $a=0,1,...,L-1$ are omitted without lose of generality.

To prove the full diversity of the proposed multiuser SF code, it is sufficient to prove the full rank of the matrices $G^{(z)}$ for all $C^{(z)} \neq C^{(z)}$. As $T^d \in T^{Q \times A_1L}$ ($Q \geq A_1L$) is a sub matrix of $G^{(z)} \in T^{N \times A_1L}$, it is adequate to prove the full rank of the matrix $T^d$, for all $C^{(z)} \neq C^{(z)}$. Moreover, it is obvious that the full rank of matrix $\tilde{T}^d$ implies the full rank of matrix $T^d$.

From the $W_tW_tL \times W_tL$ matrix given by (A7), by taking the rows given by each $l$, (i.e $l=1,...,W_L$) for all $i$, $(i=1,...,L)$, in general, we get the square matrix $T^d_1$ of size $W_tL \times W_tL$ which is the submatrix of $T^d$ of size $W_tW_tL \times W_tL$ and given by,

$$
T^d_1 = \begin{pmatrix}
\tilde{E}_{0,b,1,1} \tilde{X}^{d(z)}_{1,1} & \cdots & \tilde{E}_{L-b,1,1} \tilde{X}^{d(z)}_{1,1} \\
\vdots & \ddots & \vdots \\
\tilde{E}_{0,b,2} \tilde{X}^{d(z)}_{2,1} & \cdots & \tilde{E}_{L-b,2} \tilde{X}^{d(z)}_{2,1} \\
\vdots & \ddots & \vdots \\
\tilde{E}_{0,b,L} \tilde{X}^{d(z)}_{L,1} & \cdots & \tilde{E}_{L-b,L} \tilde{X}^{d(z)}_{L,1}
\end{pmatrix}
$$

(A8)
Note that, as $\tilde{T}_i^d \in \mathcal{T}_{W_i,L_i}^{W_t,L_t}$ is the sub matrix of $\tilde{T}_i^d$, it is obvious that the full rank of $\tilde{T}_i^d$ implies the full rank of $\tilde{T}_i^d$. This is the key condition to prove the full diversity of the proposed multiuser STF code.

The determinant of $\tilde{T}_i^d$ can be given by [5, appendix II],

$$\text{det}(\tilde{T}_i^d) = \epsilon_1 U_{1,d} + \epsilon_2 U_{2,d} \phi + \cdots + \epsilon_k U_{K,d} \phi^{K-1}, \quad (A9)$$

where, $K = (W_t - 1)L + 1$, $\epsilon_e = \pm 1$ depends on the positions of layer in the matrix $\tilde{T}_i^d$.

$$U_{1,d} = \zeta^{\tau_1} \prod_{l=1}^{\alpha_1} \tilde{\chi}_l^{(2)} \left( P_{l,d} I + n \right), \quad U_{K,d} = \zeta^{\tau_{W_t}} \prod_{l=1}^{\alpha_{W_t}} \tilde{\chi}_l^{(2)} \left( P_{l,d} I + n \right).$$

The polynomial $\zeta^{\tau_e}$ can be given by, $\tau_e = \sum_{a=0}^{K-1} \alpha_{a,e} \tau_a$.

Denoting $\tilde{T}_i = \text{diag}(\tilde{T}_1^d, \tilde{T}_2^d, \ldots, \tilde{T}_{A_u}^d)$, we have to prove,

$$\text{det}(\tilde{T}_i) = \prod_{d=1}^{A_u} \text{det}(\tilde{T}_i^d) \neq 0. \quad (A10)$$

From (A9), we have obtained,

$$\text{det}(\tilde{T}_i) = \tilde{\epsilon}_V V_1 + \tilde{\epsilon}_V V_2 \phi + \cdots + \tilde{\epsilon}_{A_u} (K-1) V_{A_u} (K-1) + 1 \Phi^{A_u (K-1)}, \quad (A11)$$

where, $V_y \in \Lambda$, $\epsilon_y \in \{1, -1\} \forall y$, $y = 1, \ldots, W_{2}$ (K-1) + 1, $V_{1} = U_{1,d}$, and $V_{A_u (K-1) + 1} = \prod_{d=1}^{A_u} U_{K,d}$.

The determinant in (A11) is taken at $S_b^{(2)} \neq S_b^{(2)}$. From (33), it can be seen that there exists at least one layer index $w (1 \leq w \leq W_{1})$ such that $S_w^{(2)} = S_w^{(2)} - S_w^{(2)} \neq 0$.

Using the fact (F1) given in [6, Appendix I], from (34), it can be obtained $S_{w}^{(2)} \left( P_{l,d} I + n \right) \neq 0 \forall_{l,n_{l,d}}$ and $l = 1, \ldots, W_{l}$, $i = 1, \ldots, W_{2}$, $n_{l,d} = 1, \ldots, W_{l}$, $d = 1, \ldots, A_u$.

Let, there are $N_0$ nonzero layers. Denoting their index in increasing order, we get $I_1, I_2, \ldots, I_{n_0}$. Following the induction given in [5, Appendix I], it follows that $\tilde{S}_{I_{n_0}} = 0$, which contradicts the assumption that, $\tilde{S}_{I_q} \neq 0 \forall_{q, q = 1, 2, \ldots, n_0}$. Then, $\text{det}(\tilde{T}_i) \neq 0$ for $\tilde{S}_{I_q} \neq 0 \forall_{q, q = 1, \ldots, N_0}$.

Thus, we have obtained that $\text{det}(\tilde{T}_i) \neq 0$ for all $S_{b_{-}} \neq 0 \quad b = 1, \ldots, B$.

We assume that there are $w$ (1 < $w$ < $Z$) out of $Z$ users have $C^{d(2)} \neq C^{d(2)}$. After zeroing some columns of (A5), $\tilde{C}^{d}$ with $w$ nonzero $\tilde{C}^{d(2)}$ can be viewed as $C^{d}$ with $Z$ nonzero $\tilde{C}^{d(2)}$. Consequently, $\tilde{C}^{d}$ with $w$ users $C^{d(2)} \neq C^{d(2)}$ has also full rank. Therefore, the proposed multiuser STF codes achieve full diversity over constellation carved from $Z[i]$.

8. References


