

# Iterative Operator-Splitting Methods for Stiff Problems in Complex Applications

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## Abstract

*In this paper we deal with the Landau-Lifshitz-Gilbert equation which describes dynamics of ferromagnetism. Based on the strong nonlinearities a stabilised discretisation method is necessary to skip the time-step restriction. In this paper we propose a implicit full discrete scheme with an embedded operator-splitting method.*

## 1. Introduction

We motivate our studying on complex models with coupled processes, e.g. transport and reaction-equations with nonlinear parameters. The ideas for decoupling the processes came from the synchronous theory in physical processes. Because of the symmetry we could decouple such processes. To be more accurate and effective a new group of methods are presented, based on higher order splitting methods, with stabilising initial-presteps. We discuss a new weighted splitting method based on a higher order iterative method and a stable first order traditional splitting method. The theory for the weighted method is presented and the applications to advanced computation with such methods are done.

## 2. Mathematical Model

The motivation for the study presented below is coming from the modelling of ferromagnetic materials, used in data storage devices, or laptop displays.

The magnetisation  $m : (0, T) \times \Omega \rightarrow S^2$  for

$$S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\} \text{ solves}$$

$$\begin{aligned} \partial_t m &= \alpha m \times (m \times (\Delta m + H(t))) \\ &+ m \times (\Delta m + H(t)) \end{aligned} \quad (1)$$

with the initial condition  $m(0) = m_0 \in W^{1,2}(\Omega; S^2)$  and the boundary condition  $\partial_n m = 0$  on  $(0, T) \times \partial\Omega$ . We have the effective outer magnetic field  $H(t) = H_{\text{eff}} + H_{\text{pulse}}(t)$ .

A simple reformation by Gilbert gives the following equation:

$$\partial_t m - \alpha m \times \partial_t m = (1 + \alpha^2) m \times (\Delta m + H(t)) \quad (2)$$

## 3. Implicit Discretisation methods

The implicit discretisation method for the equation is given below.

We apply the following notation:

$$V_h \subset W^{1,2}(\Omega, \mathbb{R}^3), \tau_h \text{ a triangulation, } m_h^0 \in V_h, \\ m_h^j \in V_h$$

The full-discretisation is given with a linearisation:

$$\begin{aligned} (\partial_t m_h^{j+1}, \phi_h)_h + \alpha (m_h^j \times \partial_t m_h^{j+1}, \phi_h)_h = \\ (1 + \alpha^2) (\bar{m}_h^{j+1/2} \times \Delta_h \bar{m}_h^{j+1/2}, \phi_h)_h \end{aligned} \quad (3)$$

where  $\phi_h \in V_h$  and  $j$  is index for the time-steps.

$$\partial_t m_h^{j+1} = k^{-1} (m_h^{j+1} - m_h^j) \text{ and } k = t^{j+1} - t^j$$

$$\bar{m}_h^{j+1/2} = 1/2 (m_h^{j+1} + m_h^j)$$

Further the constraints are given as

$$|m_h^j| = 1 \text{ discrete energy law.}$$

We deal with the full equation including the anisotropic modelling and add the magnetic field  $H_{\text{eff}}$ .

The weak formulation reads

$$\begin{aligned} (\partial_t m, \phi) + \alpha (m \times \partial_t m, \phi) = \\ (1 + \alpha^2) (m \times \Delta m, \phi) + (1 + \alpha^2) (m \times H_{\text{eff}}, \phi) \end{aligned} \quad (4)$$

with the initial condition  $m(0) = m_0 \in W^{1,2}(\Omega, S^2)$

and the magnetic field is given as  $H_{\text{eff}}$ .

For this we could also rewrite a discretisation method.

The full-discretisation is given with a linearisation:

$$\begin{aligned} & (\partial_t m_h^{j+1}, \phi_h)_h + \alpha (m_h^j \times \partial_t m_h^{j+1}, \phi_h)_h \\ (1 + \alpha^2) & \left( \overline{m}_h^{j+1/2} \times \left( \Delta_h \overline{m}_h^{j+1/2} + \overline{H}_{\text{eff}_h}^{j+1/2} \right), \phi_h \right)_h \end{aligned} \quad (5)$$

where  $\phi_h \in V_h$  and  $j$  is index for the time-steps.

#### 4. Iterative splitting method

The following algorithm is based on the iteration with fixed splitting discretization step-size  $\mathcal{T}$ , namely, on the time interval  $[t^n, t^{n+1}]$  we solve the following sub-problems consecutively for  $i = 0, 2, \dots, 2m$ , cf. [6].

$$\frac{\partial c_i(t)}{\partial t} = A c_i(t) + B c_{i-1}(t), \text{ with } c_i(t^n) = c^n \quad (6)$$

and  $c_0(t^n) = c^n$ ,  $c_{-1} = 0.0$ ,

$$\frac{\partial c_{i+1}(t)}{\partial t} = A c_i(t) + B c_{i+1}(t), \quad (7)$$

with  $c_{i+1}(t^n) = c^n$ ,

where  $c^n$  is the known split approximation at the time level  $t = t^n$ . The split approximation at the time-level  $t = t^{n+1}$  is defined as  $c^{n+1} = c_{2m+1}(t^{n+1})$ .

(Clearly, the function  $c_{i+1}(t)$  depends on the interval  $[t^n, t^{n+1}]$ , too, but, for the sake of simplicity, in our notation we omit the dependence on  $n$ .)

In the following we will analyze the convergence and the rate of the convergence of the method (6) – (7) for  $m$  tends to infinity for the linear operators  $A, B : X \rightarrow X$  where we assume that these operators and their sum are generators of the  $C_0$  semigroups. We emphasize that these operators aren't necessarily bounded, so, the convergence is examined in general Banach space setting.

**Theorem 1.** *Let us consider the abstract Cauchy problem in a Banach space  $X$*

$$\begin{aligned} \partial_t c(t) &= A c(t) + B c(t), & 0 < t \leq T \\ c(0) &= c_0 \end{aligned} \quad (8)$$

where  $A, B, A + B : X \rightarrow X$  are given linear operators being generators of the  $C_0$ -semigroup and  $c_0 \in X$  is a given element. Then the iteration process (6)–(7) is convergent and the rate of the convergence is of second order.

**Remark 1.** When  $A$  and  $B$  are matrices (i.e. (6)–(7) is a system of the ordinary differential equations), for the growth estimation we can use the concept of the logarithmic norm. (See e.g.[13].) Hence, for many important class of matrices we can prove the validity.

**Remark 2.** We note that a huge class of important differential operators generate contractive semigroup. This means that for such problems - assuming the exact solvability of the split sub-problems - the iterative splitting method is convergent in second order to the exact solution.

Decoupling of the Landau-Lifschitz-Gilbert equation:

$$u_t = u \times \Delta u - \lambda u \times (u \times \Delta u) \quad (9)$$

where the operators are given as

$$A(u) = u \times \Delta u \quad (10)$$

$$B(u) = -\lambda u \times (u \times \Delta u) \quad (11)$$

where  $A(u)$  is the stabilisation or non-stiff operator which could be discretised with implicit methods and  $B(u)$  is the stiff operator which destabilise the system.

First ideas are done in the work of [Geiser et al 2006].

In the next subsection we present the stability theory of the LLG-equation.

#### 5. Stability Theory for the LLG-equation

In the works of [1] and [15] the problems of a blow up for the gradients of the Landau-Lifschitz-Gilbert equations are discussed and stabilisation methods are given.

Here we concentrate on a special nonlinear case  $H(u)$  and time-dependent case  $H(t)$ .

For both we could derive a stable implicit discretisation method. The algorithm is given as

Algorithm: Fixpoint-algorithm to solve 5

**Algorithm**

Set  $m_h^{j+1,1} := m_h^j$  and  $l := 0$

(i) Compute  $m_h^{j+1,l+1} \in V_h$  such that for all  $\phi_h \in V_h$  there holds:

$$\begin{aligned} 1/k & (m_h^{j+1,l+1}, \phi_h)_h + \alpha/k (m_h^j \times m_h^{j+1,l+1}, \phi_h)_h \\ & - (1 + \alpha^2)/4 \left( \overline{m}_h^{j+1,l+1} \times \left( \Delta_h \overline{m}_h^{j+1,l+1} + H(m_h^{j+1,l}) \right), \phi_h \right)_h \end{aligned} \quad (12)$$

$$- (1 + \alpha^2)/4 \left( \overline{m}_h^j \times \left( \Delta_h \overline{m}_h^{j+1,l} + H(m_h^{j+1,l}) \right), \phi_h \right)_h$$

$$- (1 + \alpha^2)/4 \left( \overline{m}_h^j \times \left( \Delta_h \overline{m}_h^{j+1,l+1} + H(m_h^{j+1,l+1}) \right), \phi_h \right)_h$$

$$= 1/k (m_h^j, \phi_h)_h +$$

$$(1 + \alpha^2) / 4 \left( \bar{m}_h^j \times (\Delta_h \bar{m}_h^j + H(m_h^j)), \phi_h \right)_h$$

(ii) If  $\|m_h^{j+1,1} - m_h^{j+1,l}\|_h \leq \varepsilon$  then stop and set

$$m_h^{j+1} := m_h^{j+1,1}.$$

(iii) Set  $l := l + 1$  and go to (i).

We can derive the error estimates:

**Lemma 1.** *Suppose that we have  $\gamma$  dependent from  $h, k$  and  $\alpha$  and  $|m_h^j(x_m)| = 1$  for all  $m$ . Then for all  $l \geq 1$  there holds*

$$\|m_h^{j+1,1} - m_h^{j+1,l}\|_h \leq \Theta \gamma \|m_h^{j+1,1} - m_h^{j+1,l-1}\|_h. \quad (13)$$

*Proof.* The proof can be done with respect to the ideas of [1], while the function  $H(u)$  and  $H(t)$  are linear functions, we could extend our lemma.

## 6. Numerical Results

Our numerical examples are based on the magnetisation of a isotropic and anisotropic material.

The isotropic material on the one hand is hard to control, where the anisotropic material has special directions to follow by the discretisation.

We start with a first example to use the stable first order splitting as a pre-step method and then start with the higher order iterative method.

### 6.1. Numerical Experiments (pure isotropic case)

We start with the pure isotropic case, not influenced by an outer magnetic field.

$$u_t = u \times \Delta u - \lambda u \times (u + \Delta u) \quad (14)$$

where  $\lambda = 1$ ,  $\Omega = [-0.5, 0.5]^2$ .

We concentrate on different initial values that complicate the computations:

Initial values

1. Case

$$u_0 = \begin{cases} (2xA; A^2 - |x|^2) / (A^2 + |x|^2) & \text{for } |x| < 0.5 \\ (0; 0; -1) & \text{for } |x| > 0.5 \end{cases} \quad (15)$$

2. Case

$$u_0 = \begin{cases} (2xA; A^2 - |x|^2) / (A^2 + |x|^2) & \text{for } |x| < 0.05 \\ (0; 0; -1) & \text{for } |x| > 0.05 \end{cases} \quad (16)$$

3. Case

$$u_0 = [2xA; A^2 - |x|^2] / (A^2 + |x|^2) \text{ for } x \in \Omega \quad (17)$$

where  $A = (1 - 2|x|)^4 / 4$

The convergence-results are given in the table:

**Table 1.** Relative errors between  $l = 4$  and  $l = 6$

x	y	$L_2$ (relative error)	$L_\infty$ (relative error)
0	0	0.487524	1.999790
-1	0	0.201978	0.426290
0	-1	0.199584	0.426297
1	0	0.198766	0.426282
0	1	0.201194	0.426287

**Table 2.** Relative errors between  $l = 5$  and  $l = 6$

x	y	$L_2$ (relative error)	$L_\infty$ (relative error)
0	0	0.662921	1.999939
-1	0	0.191115	0.453568
0	-1	0.194658	0.468610
1	0	0.198813	0.493772
0	1	0.195291	0.479565

### 6.2. Second test-example with LLG with anisotropy $H(t)$

In a next example we start with an anisotropy field, coming from the outer magnetic field.

Our equations are given as

$$u_t = u \times \Delta u - \lambda u \times (u \times (\Delta u + H_{\text{eff}})) \quad (18)$$

where  $\lambda = 1$ ,  $\Omega = [-0.5, 0.5]^2$ .

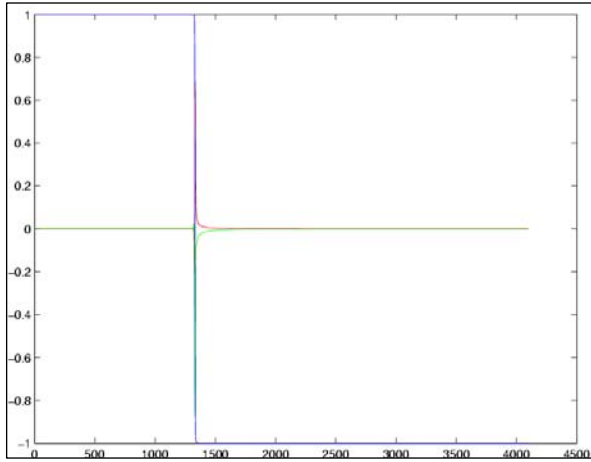
For a experiment, we concentrate on different initial values that complicate the computations and we deal with the following initial values:

$$u_0 = \begin{cases} (2xA; A^2 - |x|^2) / (A^2 + |x|^2) & \text{for } |x| < 0.5 \\ (0; 0; -1) & \text{for } |x| > 0.5 \end{cases} \quad (19)$$

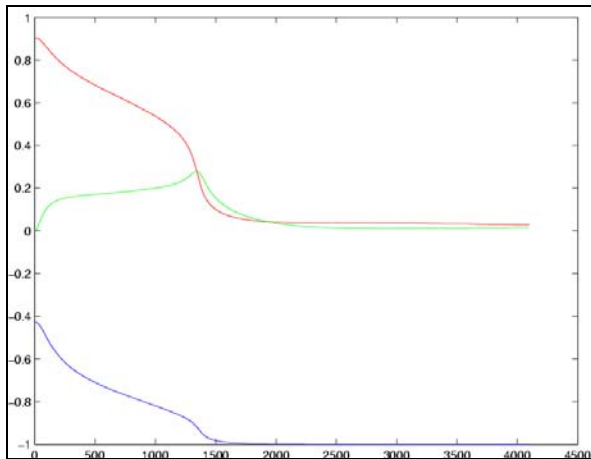
Our effective magnetic field is given as:

$$H_{\text{eff}} = (0.01, 0.01, 0.01)^T \text{ for all } x \in \Omega. \quad (20)$$

We consider special point with respect to their relaxation. In the next figure 1 we obtain the relaxation at the points (0, 0) (singular point) and (0.125, 0) (non-singular point). For the numerical examples we use the timestep  $\Delta t = 2^{-12} / 10$



**Figure 1.** Magnetisation direction of the magnetisation-vector in point (0,0), where the colors are the coordinates of the vector : red = x, green = y, blue = z.



**Figure 2.** Magnetisation direction of the vector in point (0.125,0), where the colors are the coordinates of the vector : red = x, green = y, blue = z.

and the spatial step  $\Delta x = 1/2^7$ . As a result we obtain that the magnetisation in the field is relaxed from a non-equilibrium state to an equilibrium state.

As a further result we see that the effective magnetic field influence the singular point, cf. figure 1.

In the fast scales the relaxation is given in a singular point, see fig. 1. On the other hand slow relaxation are given in figure 2. There we found out that the effective field slow down the relaxation.

Due to our modelling interest, such problematic singular points are difficult to control, where the non-singular points are better understood.

This effects may help to understand the switching in the different regions of a magnetic material.

## 7. Conclusions and Discussions

We present the combination with isotropic and anisotropic magnetisation behaviour. Stable discretisation methods based on penalising methods are described, further splitting methods could decouple the complicate equations. In numerical experiments we verify our theoretical results. The experiments present the different situations of singular and non-singular regions, where in the non-singular regions a magnetic switch is better to control. In future the nonsmooth treatment of magnetic fields with comparison of experimental dates are proposed.

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