Chaotification of Linear Dynamical System via Impulsive Control

Jin Huan\textsuperscript{1}, Li Chuandong\textsuperscript{2}

\textsuperscript{1}Jining Vocational and Technical College, Shandong, 272037
\textsuperscript{2}Computer College of Chongqing University, Chongqing, 400044
jh6902@126.com, licd@cqu.edu.cn
doi: 10.4156/ijipm.vol2.issue3.4

Abstract

This paper is concerned with chaotification of linear dynamical system via impulsive control. Some sufficient conditions are established to guarantee impulsive system have chaos by using discrete dynamical system in linear impulsive system. Examples and simulations are given to verify the effectiveness of the theoretical results.

Keywords: Chaotification, Impulsive Control

1. Introduction

Many systems existing in physics, chemistry, biology, engineering and information have impulsive dynamical behavior due to abrupt jumps at certain instants during the evolving processes, which can be modeled by impulsive differential equations \cite{1, 2, 12}. Recent years we have seen wide-scope potential applications of IDEs in various scientific fields, such as impulsive control \cite{17}, impulsive synchronization \cite{18}, etc. Recent years, research on chaotification (chaos anticontrol) has seen in rapid evolution and extension toward engineering applications. Chaotifying discrete-time system has been rigorously developed in \cite{3, 4}. Liu and Guan \cite{13, 14} chaotify discrete system via impulsive control. Unlike the discrete case, however, chaotification in continuous flows is very complicated and difficult. Chen and Yang \cite{5} chaotify a continuous system near stable limit cycle. Yang \cite{16} and Jiang \cite{10} chaotify a continuous nonlinear system via impulsive input. Shu \cite{15} chaotified a linear hyperbolic system of partial differential equations by means of nonlinear boundary reflection. Chaotification of some specific linear system via impulsive control was investigated in \cite{6, 7, 11}.

In this paper, a linear system, which dynamics is very simple, is used to build a linear impulsive system. The purpose of this letter is to find the condition of chaos theoretically by using discrete dynamical system in linear impulsive system. The organization of this paper is as follows. A linear system with impulses at fixed time is introduced and a discrete dynamical system is obtained and the conditions of existence of chaos are discussed in section 2. In section 3, numerical results, such as time-series plot, chaotic attractors are given by some examples. Finally, in section 4 we present a brief conclusion.

2. Chaotification

Consider a linear dynamical system in the form of

$$\dot{x} = Ax$$

(1)

Where $x \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times m}$ is a constant matrix and $x(0) = x_0$ is the initial conditions. The solution of system (1) is $x(t) = e^{At}x_0$, $e^{At}$ is fundamental matrix of system (1).

In the sequel, we study system (1) with impulses at periodic fixed time

* This research was supported by Natural Science Foundation of China (60974020) and the Aid Financially of Visiting Scholar Project of Shandong in China.
Chaotification of Linear Dynamical System via Impulsive Control
Jin Huan, Li Chuandong
International Journal of Information Processing and Management Volume 2, Number 3, July 2011

\[
\begin{cases}
\dot{x} = Ax, & t \neq nT \\
x(nT^+) = f(x(nT)), & t = nT \\
x(0^+) = x_0, & t = 0
\end{cases}
\] (2)

where \( x(nT^+) = \lim_{\varepsilon \to 0^+} x(nT + \varepsilon) \), \( f : \mathbb{R}^m \to \mathbb{R}^m \) is continuous nonlinear map.

We shall discuss the chaotic dynamics of system (2). In the case of system (1) subject to an impulsive force, the external impulsive force will make the displacement \( x(nT) \) change to \( x(nT^+) \) instantaneously at \( t = nT \). We view \( T \) as a parameter in system (2).

**Theorem 1.** Let \( \phi(t) = f\left(e^{AT} x_t\right) \), if \( \phi : \Omega \to \Omega, \Omega \in \mathbb{R}^m \) is chaotic. Then system (2) is chaotic on \( \Omega \).

**Proof.** We first calculate the solution of system (2) step by step.

(i) For \( t \in [0, T) \), the equation \( \dot{x} = Ax \) and initial date \( x(0^+) = x_0 \) implies that
\[
x(t) = e^{tA} x_0, t \in [0, T), \text{ thus } x(T) = e^{AT} x_0.
\]
The impulsive control switches \( x(T) \) to \( x(T^+) = f(x(T)) = f(e^{AT} x_0) = g(x_0) \). Thus, for \( t \in [T, 2T) \), the system has initial state as \( x(T^+) = g(x_0) \).

(ii) Now for \( t \in [T, 2T) \), the equation \( \dot{x} = Ax \) and initial state \( x(T^+) = g(x_0) \) implies that
\[
x(t) = e^{(t-T)A} g(x_0), t \in [T, 2T), \text{ then } x(2T) = e^{AT} g(x_0).
\]
The impulsive control switches \( x(2T) \) to \( x(2T^+) = f(x(2T)) = f(e^{AT} g(x_0)) = g^2(x_0) \).

(iii) Repeated this proposes we get the solution of system (2) as

\[
\begin{align*}
0 \leq t < T & \quad x(0^+) = x_0, x(t) = e^{AT} x_0 \\
T \leq t < 2T & \quad x(T^+) = g(x_0), x(t) = e^{(t-T)A} x(T^+) = e^{(t-T)A} g(x_0) \\
2T \leq t < 3T & \quad x(2T^+) = g^2(x_0), x(t) = e^{(t-2T)A} x(2T^+) = e^{(t-2T)A} g^2(x_0) \\
& \vdots \\
nT \leq t < (n+1)T & \quad x(nT^+) = g^n(x_0), x(t) = e^{(t-nT)A} x(nT^+) = e^{(t-nT)A} g^n(x_0)
\end{align*}
\] (3)

System (3) is a set of “hair” with roots \( \left\{ g^n(x_0) : n \in \mathbb{N} \right\} \), if \( g \) is chaotic then \( \left\{ g^n(x_0) : n \in \mathbb{N} \right\} \) is a chaotic sequence, thus the set of “hairs” system (3) is chaotic.

**Remark 1.** From theorem 1, when \( g : \Omega \subseteq \mathbb{R}^m \to \Omega \) is chaotic, \( e^{AT} x_0 \) is in \( \Omega \). System (2) can be chaotic only when \( T \) has special value for one given matrix \( A \).

### 3. Examples and Simulations

**Example 1.** Consider a one-dimensional linear impulsive dynamical system
Let \( g \) is a logistic map: 
\[
g(x) = \mu x(1-x), \quad 0 \leq x \leq 1.
\]
When \( \mu = 4 \), \( g \) is chaotic and the bound of \( g \) is \([0,1]\). Then 
\[
f(x) = g(e^{-ax} x) = 4e^{-ax} x(1-e^{-ax}).
\]
Thus if \( a, T \) are chosen so that 
\[
e^{-ax} \leq 1.
\]
For \( a \in R^+ \), given any number \( T > 0 \), the system (4) is chaotic on \([0,1]\). Let \( a = 1, x_0 = 0.9058 \). The time-series of \( x \) of system (4) with \( T = 0.5 \) is given in Figure 1. Figure 2 shows the chaotic attractor of system (4). A modified fourth-order Runge-Kutta method with step size \( 10^{-3} \) is applied to solve (4) numerically.

**Example 2.** Consider a two-dimensional linear impulsive dynamical system

\[
\begin{align*}
\dot{x} &= Ax \\
x(nT^+) &= f(x(nT)) \\
x(0^+) &= x_0
\end{align*}
\]

Let \( g \) is a lozi map [9]: 
\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = g\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 1-a|x_1| + bx_2 \\ x_1 \end{pmatrix}.
\]

From [9], we know that \( g \) is chaotic when \( a = 1.7, b = 0.5 \). Let further \( x_0 = (0.9575, 0.9649)' \), \( A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), and 
\[
f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = g\left(e^{-ax} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 1-a\exp(-T)x_1 + b\exp(T)x_2 \\ \exp(-T)x_1 \end{pmatrix}.
\]

Let \( T = 0.5 \), Figure 3 shows a periodic attractors of system (5). Let \( T = 1 \), Figure 4 shows a chaotic attractors of system. Now, we let 
\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = g\left(e^{-ax} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 1-a\exp(-T)x_1 + b\exp(-T)x_2 \\ \exp(-T)x_1 \end{pmatrix}.
\]

Then the system (5) exhibits a quasi-periodic attractor when \( T = 0.6 \) (as shown in Figure 5), and a chaotic attractor when \( T = 1 \) (as shown in Figure 6).

**Example 3.** Consider a three-dimensional linear impulsive dynamical system

\[
\begin{align*}
\dot{x} &= Ax \\
x(nT^+) &= f(x(nT)) \\
x(0^+) &= x_0
\end{align*}
\]

Let \( g \) is a three-dimensional generalization Henon map [8]:
Chaotification of Linear Dynamical System via Impulsive Control
Jin Huan, Li Chuandong
International Journal of Information Processing and Management Volume 2, Number 3, July 2011

\[
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix}
= g
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix}
= \begin{pmatrix}
    -\alpha x^2 + y + 1 \\
    \beta x + z \\
    -\beta x
\end{pmatrix}.
\]

When \( \alpha = 1.07, \beta = 0.3 \), \( g \) is chaotic. Let \( x_0 = (0.7577, 0.7431, 0.3922)' \),

\[
A = \begin{bmatrix}
    1 & 2 & 0.5 \\
    2 & 0.5 \\
    0.5
\end{bmatrix},
\]

\[
f \left( \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} \right) = g
\left( \begin{bmatrix}
    e^{-AT} x_1 \\
    x_2 \\
    x_3
\end{bmatrix} \right) = \begin{pmatrix}
    -\alpha e^{-2T} x_1^2 + e^{-2T} x_2 + 1 \\
    \beta e^{-T} x_1 + e^{-0.5T} x_3 \\
    -\beta e^{-T} x_1
\end{pmatrix},
\]

Then system (6) has a chaotic attractor when \( T = 0.5 \), as shown in Figure 7. When

\[
A = \begin{bmatrix}
    -1 & 2 & 0.5 \\
    2 & 0.5 \\
    0.5
\end{bmatrix},
\]

\[
f \left( \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} \right) = g
\left( \begin{bmatrix}
    e^{-AT} x_1 \\
    x_2 \\
    x_3
\end{bmatrix} \right) = \begin{pmatrix}
    -\alpha e^{2T} x_1^2 + e^{2T} x_2 + 1 \\
    \beta e^T x_1 + e^{-0.5T} x_3 \\
    -\beta e^T x_1
\end{pmatrix},
\]

\( T = 0.5 \), a chaotic attractor also occurs, as shown in Figure 8.

4. Conclusion

Chaotification of linear dynamical system via impulsive control has been investigated in this paper. A chaotification theorem has been established to determine whether linear impulsive system has chaotic behavior. Several examples and simulations have also presented to verify the effectiveness of the theoretical results.

Figure 1: The time-series of \( x \) of system (4) with \( T = 0.5 \)
Figure 2. The chaotic attractors of system (4) with $T = 0.5$.

Figure 3. The periodic attractors of system (5) with $T = 0.5$.

Figure 4. The chaotic attractors of system (5) with $T = 1$. 
Figure 5. The quasi-periodic attractors of system (5) with $T = 0.6$

Figure 6. The chaotic attractors of system (5) with $T = 1$

Figure 7. The chaotic attractors of system (6) with $T = 0.5$

Figure 8. The chaotic attractors of system (6) with $T = 0.5$
5. References


