A Novel Filled Function for Solving Non-smooth Global Optimization Problem

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Abstract

This paper presents a new filled function for identifying global optimizers or approximate global optimizers of the non-smooth unconstrained global minimization problem. The proposed filled function contains only one parameter whose value can be adjusted easily at each iteration. The theoretical properties of the new filled function are also investigated. Based on the filled function theory, a corresponding solution algorithm is proposed. Numerical results from some test functions demonstrate that our filled function approach is promising.

Keywords: Non-smooth Unconstrained global Optimization, Filled Function, Filled Function Method, Local Minimizer, Global Minimizer.

1. Introduction

Optimization of a general cost function arises frequently in various applications such as scheduling, design and operation problems (see [11,12]). The literatures on optimization can be divided into two classes: local optimization approaches and global optimization approaches. Even though global optimization problems have been studied since 1960’s, most of researchers then still paid attention to the local optimization approaches. The impacts of global optimization have become influential only in the last few decades due to the advancement of computer technologies and the increasing dependence on the need to search for the global optimization solutions from the real world applications. Since 1970’s, many new theories and algorithms on global optimization are emerging. In particular, the filled function method introduced by Ge and Qin in [1] is a practical and useful tool for global optimization. The filled function method contains two phases: local minimization and filling procedure. The first phase aims to search for a local minimizer of the objective function. The second phase is used to find a better initial point for the first phase. The two phases are performed alternatively until the preset terminating criteria is satisfied and an approximate global minimizer is obtained.

Since Ge and Qin presented the filled function method, rapid progress has been made both in theories and practical algorithms for the filled function method. See, for example, papers [2-7]. Note that the above filled functions are used mainly for smooth global optimization. In practice, however, many problems allow objective functions to be non-smooth. In this paper, we extend the filled function method for smooth global optimization to the case in which the objective function is non-smooth, and construct a new filled function. The new filled function contains only one parameter and can be easily adjusted. A numerical experimentation is also performed and its computational results are reported.

Generally speaking, there are two difficulties in global optimization faced by us: one issue is how to leave from the current local minimizer to find a better one; the another is how to check the current minimizer is a global one. Our paper deals only with the first issue.

The rest of the paper is organized as follows: In Section 2, we will present some basic knowledge on non-smooth analysis and optimization, and make some assumptions on the objective function. In Section 3, we will propose a new filled function and discuss its properties. In Section 4, we will establish a corresponding filled function algorithm. In Section 5, we will report preliminary numerical results, and in Section 6, we give our conclusions.
2. Basic knowledge

Consider the following non-smooth global minimization problem

\((P_o) : \min_{x \in \mathbb{R}^n} f(x), \) where \(f : \mathbb{R}^n \rightarrow \mathbb{R}.\)

In this section, we give some basic theoretical results about non-smooth analysis from \([10]\), make some assumptions on objective function and introduce the definition of filled function for \((P_o).\)

Definition 2.1: Suppose that \(f\) is Lipschitz with constant \(L > 0\) at the point \(x,\) the generalized gradient of \(f\) at \(x\) is declared to be

\[\partial f(x) = \{\xi \in \mathbb{R}^n : \xi^T d \leq f^0(x,d), \forall d \in \mathbb{R}^n\},\]

where

\[f^0(x,d) = \limsup_{y \to x, t \downarrow 0} \frac{f(y + td) - f(y)}{t}.\]

Lemma 2.1: Let \(f\) be Lipschitz with constant \(L > 0\) at the point \(x,\) then

(a) \(f^0(x,d)\) is finite, and satisfies \(|f^0(x,d)| \leq L \|d\|;\)

(b) \(\partial \sum s_i f_i(x) \subseteq \sum s_i \partial f_i(x),\) for \(\forall s_i \in \mathbb{R}^1.\)

(c) \(\partial f(x)\) is a nonempty bounded and closed convex set, and it holds \(\|\xi\| \leq L,\) for \(\forall \xi \in \partial f(x).\)

To proceed, we make the following assumptions throughout this paper.

Assumption 1. \(f(x)\) is Lipschitz continuous on \(\mathbb{R}^n\) with Lipschitz constant \(L > 0.\)

Assumption 2. \(f(x)\) is coercive, that is, \(\lim_{x \to \pm \infty} f(x) = +\infty.\)

Notes that Assumption 2 implies the existence of a bounded box set \(X \subset \mathbb{R}^n\) whose interior contains all minimizers of \(f(x).\) We assume that the value of \(f(x)\) for any \(x\) on the boundary of \(X\) is greater than the value of \(f(x)\) for \(x\) in the interior, then the problem \((P_o)\) can be reduced into the following equivalent problem formulation \((P): \min_{x \in X} f(x).\)

Assumption 3. The problem \((P)\) has at least one global minimizer and has only a finite number of different minimums over \(X.\)

Let \(x^*\) be a local minimizer of \(f(x).\) Now, we introduce the definition of filled function for the non-smooth global minimization problem \((P),\) which will be used in this paper.

Definition 2.2: A function \(P(x, x^*)\) is said to be a filled function of \(f(x)\) at \(x^*,\) if it satisfies the following conditions:

1. \(f(x)\) is a strict maximizer of \(P(x, x^*)\) over \(X.\)
2. It holds \(0 \notin \partial P(x, x^*)\) for any \(x \in S_1 = \{x \in X : x \neq x^*, f(x) \geq f(x^*)\}.\)
3. If \(x^*\) is not a global minimizer, then \(P(x, x^*)\) does have one minimizer in the region \(S_2 = \{x \in X : f(x) < f(x^*)\}.\)

3. A new filled function and its properties

For convenience, we denote the set of local minimizers for problem \((P)\) by \(L(P),\) the set of global minimizers by \(G(P),\) and \(M = \max_{x, y \in X} \|x - y\|.\) Let \(x^* \in L(P).\) Now, we present a new filled
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function with one parameter for problem \((P)\) as follows:

\[
F(x, x^*, r) = (f(x) - f(x^*) + r) \exp\left(\frac{1}{r^\alpha (r + \|x - x^*\|)}\right),
\]

where \(r\) is an adjustable parameter satisfying \(0 < r < f(x^*) - f(x_g)\), where \(x_g \in G(P)\), \(\alpha > 1\) is a constant, and \(\|\|\) indicates the Euclidean vector norm.

The following theorems show that \(F(x, x^*, r)\) is a filled function if \(r > 0\) is small enough.

**Theorem 3.1.** Let \(x^* \in L(P)\). If \(r\) satisfies \(0 < r < \min(1, (L(M + 1))^{\frac{2}{1-\alpha}})\), then \(x^*\) is a strict local maximizer of \(F(x, x^*, r)\).

Proof. Since \(x^*\) is a local minimizer of \(f(x)\), there exists a neighborhood \(N(x^*, \sigma)\) of \(x^*\) with \(\sigma > 0\) such that \(f(x) \geq f(x^*)\) for all \(x \in N(x^*, \sigma) \cap X\). By Assumption 1, for any \(x \in N(x^*, \sigma) \cap X \setminus x^*\), we have

\[
F(x, x^*, r) = (L\|x - x^*\| + r) \exp\left(\frac{1}{r^\alpha (r + \|x - x^*\|)}\right) \tag{1}
\]

Let \(g(t) = (Lt + r) \exp\left(\frac{1}{r^\alpha (r + t)}\right)\), where \(0 \leq t \leq M\). For \(0 < r < \min(1, (L(M + 1))^{\frac{2}{1-\alpha}})\), we have that

\[
g'(t) = \exp\left(\frac{1}{r^\alpha (r + t)}\right)[L - (Lt + r) \frac{1}{r^\alpha (r + t)^2}] \leq \exp\left(\frac{1}{r^\alpha (r + t)}\right)[L - r \frac{1}{r^\alpha (r + M)^2}]
\]

\[
\leq \exp\left(\frac{1}{r^\alpha (r + t)}\right)[L - (Lt + r) \frac{1}{r^\alpha (r + t)^2}] \leq \exp\left(\frac{1}{r^\alpha (r + t)}\right)[L - \frac{1}{r^\alpha (1 + M)^2}] < 0 \tag{2}
\]

It follows from (1) and (2) that, for \(x \neq x^*\),

\[
F(x, x^*, r) \leq g(\|x - x^*\|) < g(0) = F(x^*, x^*, r).
\]

Hence, \(x^*\) is a strict local maximizer of \(F(x, x^*, r)\).

**Theorem 3.2.** Let \(x^* \in L(P)\) and \(x \in S_t\). If \(r\) satisfies \(0 < r < \min(1, (L(M + 1))^{\frac{2}{1-\alpha}})\), then \(0 \notin \partial F(x, x^*, r)\).

Proof: For any \(x \in S_t\), that is, \(f(x) \geq f(x^*)\) and \(x \neq x^*\), if the parameter \(r\) satisfies

\[
0 < r < \min(1, (L(M + 1))^{\frac{2}{1-\alpha}}),
\]

then we have

\[
\left\langle \frac{x - x^*}{\|x - x^*\|}, \partial F(x, x^*, r) \right\rangle \leq \exp\left(\frac{1}{r^\alpha (r + \|x - x^*\|)}\right)\left[\frac{x - x^*}{\|x - x^*\|}, \partial f(x)\right] - \frac{f(x) - f(x^*) + r}{r^\alpha (r + \|x - x^*\|)^2}
\]

\[
\leq \exp\left(\frac{1}{r^\alpha (r + \|x - x^*\|)}\right)[L - \frac{r}{r^\alpha (r + \|x - x^*\|)^2}] \leq \exp\left(\frac{1}{r^\alpha (r + \|x - x^*\|)^2}\right)[L - \frac{r}{r^\alpha (1 + M)^2}] < 0.
\]

Therefore, \(0 \notin \partial F(x, x^*, r)\).
Theorem 3.3. Suppose that $x^*$ is a local minimizer but it is not a global minimizer of $f(x)$, then there exists a minimizer $x_0$ of $F(x, x^*, r)$ which lies in the set $S_2$.

Proof. By the given condition, there exist $r > 0$ and a point $x'$ with $f(x') < f(x^*)$, such that $f(x') - f(x^*) + r < 0$.

Let $\partial X$ stand for the boundary of the box set $X$. Since $f(x)$ is coercive, the value of $f(x)$ on the $\partial X$ is larger than the function value in its interior. Therefore, for any $x \in \partial X$, we have $F(x, x^*, r) > 0$.

Suppose that the function $F(x, x^*, r)$ over $X$ attains its minimum at $x_0$. When $r > 0$ is small enough, it follows from the above inequality that

$$\min_{x \in X} F(x, x^*, r) = \min_{x \in X; x \in \partial X} F(x, x^*, r) = F(x_0, x^*, r) \leq F(x', x^*, r) < 0.$$

Since $X \setminus \partial X$ is an open set, we have $x_0 \in X \setminus \partial X$ and $f(x_0) < f(x^*)$, which implies $x_0 \in S_2$.

Theorem 3.4. Suppose that $x_1, x_2 \in X$ satisfy the following conditions:

$$\min(f(x_1), f(x_2)) \geq f(x^*) \left\| x_2 - x^* \right\| > \left\| x_1 - x^* \right\| + \epsilon, \text{ where } \epsilon > 0,$$

then, when $r > 0$ is appropriate small, it holds $F(x_1, x^*, r) > F(x_2, x^*, r)$.

Theorem 3.5. Suppose that $x_1, x_2 \in X$ satisfy the following conditions:

$$f(x_2) \geq f(x^*) > f(x_1), f(x_1) - f(x^*) + > 0, \left\| x_2 - x^* \right\| > \left\| x_1 - x^* \right\| + \epsilon, \text{ where } \epsilon > 0,$$

then, when $r > 0$ is appropriate small, it holds $F(x_2, x^*, r) > F(x_1, x^*, r)$.

We omit the proof of both Theorem 3.4 and Theorem 3.5 since they are similar to that for Theorem 3.1.

4. Solution algorithm

In the above section, we investigated the theoretical properties of the filled function, we are now in a position to present the corresponding algorithm.

Filled function algorithm:

1. Initialization step

Choose $\epsilon_0 = 10^{-6}$ as a tolerance parameter for terminating the minimization process; Choose an initial point $x_i \in X$; Choose $\epsilon_k$, $i = 1, 2, \ldots, k_0$ with $k_0 > 2n$, where $n$ is the number of variables; Set $k = 1$, and go to the main step.

2. Main step

Step 1. Obtain a local minimizer $x_i^*$ of $(P)$ by implementing any non-smooth local minimization procedure, starting from the initial point $x_i$ and go to Step 2.

Step 2. Let $r = 1$ and go to Step 3.

Step 3. If $k \leq k_0$, then set $x^- = x_i^* + 0.1\epsilon_k$, construct the filled function $F(x, x^*, r)$ and use $x^-$ as an initial point for the local minimization of the following problem: $\min_{x \in X} F(x, x^*, r)$. Let $x_k$ be its local minimizer. If $x^-$ arrives at the boundary of $X$ during its minimization process, then set $k = k + 1$, and repeat Step 3; Otherwise, go to Step 4.
Step 4. If all the following conditions are satisfied, then set \( x^- = x_k, k = 1 \), and use \( x^- \) as an initial point to get another local solution \( x^{*}_2 \) of \((P)\), and go to Step 5; Otherwise, go to Step 6.

(a): \( f(x^-) < f(x^*_1) \);  
(b): \( F(x_k, x^*_1, r) > F(x_{k-1}, x^*_1, r) \);  
(c): \( (x_k - x^*_1)^T \xi \geq 0, \forall \xi \in \partial F(x_k, x^*_1, r) \);  
(d): \( \| \xi \| < 10^{-4}, \forall \xi \in \partial F(x_k, x^*_1, r) \).

Step 5. If \( f(x^-) < f(x^*_1) \), then set \( x^*_1 = x^*_2 \), and go to Step 2; Otherwise, go to Step 6.

Step 6. Set \( r = 0.1r \). If \( r \geq r^* \), then set \( k = 1 \), and go to Step 2; Otherwise, the algorithm stops and \( x^*_1 \) is taken as a global minimizer.

Now, we make some remarks on the above algorithm.

1. The proposed algorithm can be used to solve smooth global optimization problem as well.
2. The choice of the direction vectors \( e_i = (e_i(1),...,e_i(n)) \) can be defined as follows.

\[
e_i(1) = \sin(\theta_{i1}) \sin(\theta_{i2})...\sin(\theta_{in-1}) \sin(\theta_{in}),
\]

\[
e_i(2) = \sin(\theta_{i1}) \cos(\theta_{i2})...\sin(\theta_{i,n-1}) \cos(\theta_{in-1}) \cos(\theta_{in}),
\]

\[
e_i(n-1) = \sin(\theta_{i1}) \cos(\theta_{i2})...\sin(\theta_{i,n-1}) \cos(\theta_{in-1}) \cos(\theta_{in}),
\]

\[
e_i(n) = \cos(\theta_{i1}) \cos(\theta_{i2})...\sin(\theta_{i,n-1}) \cos(\theta_{in-1}) \cos(\theta_{in}),
\]

where \( n \) is the number of variables and
\[
\theta_{ij} \in \left\{ \frac{k\pi}{16} : k = 1,...,32 \right\}, j = 1,...,n-1, i = 1,...,k_i.
\]

3. In the filled function algorithm, the most important thing is to minimize \( f(x) \) or \( F(x, x^*, r) \) via applying a certain local minimization procedure on them. The following non-smooth local minimization procedures can be used for this propose: Hybrid Hooke and Jeeves-Direct Method for Non-smooth Optimization [9], Mesh Adaptive Direct Search Algorithms for Constrained Optimization [8], Bundle methods, Powell’s method, etc.

4. The method contains two phases: local minimization and filling. The second phase aims to search for an improved initial point for the first phase. So, once a point \( x^- \in X \) with \( f(x^-) < f(x^*_1) \) is found during the phase of filling, the minimization of \( F(x, x^*, r) \) stops and filled function algorithm turns to the minimization of \( f(x) \).

5. Numerical experiment

To evaluate the computational performance of our algorithm, in this section, we tested it on several functions. The proposed algorithm is programmed in Fortran 95 for working on the Windows XP system. In non-smooth case, we use the Hybrid Hooke and Jeeves-Direct Method to search for a local minimizer. In smooth case, we apply the BFGS Method to find a local minimizer.

**Problem 1.** \( \min f(x) = \left| \frac{x-1}{4} \right| + \left| \sin(\pi(1+\frac{x-1}{4})) \right| + 10 \), \( s.t. \ |x| \leq 10 \).

The algorithm successfully located a global solution: \( x^* = 1 \) with \( f(x^*) = 10 \). Table 1 records the numerical results of Problem 1.

**Problem 2.** \( \min f(x) = \max \{ 5x_1 + x_2, -5x_1 + x_2, x_1^2 + x_2^2 + 4x_2 \} \), \( s.t. \ |x_1| \leq 4, |x_2| \leq 4 \).

The algorithm successfully found a global solution: \( x^* = (0,-3)^T \) with \( f(x^*) = -3 \). Table 2 records the numerical results of Problem 2.
The algorithm successfully found a global solution: $x^* = (1, 0.5, ..., 0.1)^T$ with $f(x^*) = 0$. Table 3 records the numerical results of Problem 3.

Problem 4. $\min f(x) = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 - x_1x_2 - 4x_2^2 + 4x_2^4$, s.t. $|x_1| \leq 3, |x_2| \leq 3$.

The algorithm successfully found a global solution: $x^* = (-0.0898, -0.7126)^T$ with $f(x^*) = -1.0316$. Table 4 records the numerical results of Problem 4.

The meanings of the symbols used in the tables are explained as follows:
- $k$: The iteration number in finding the $k$th local minimizer.
- $r$: The parameter to find the $k$th local minimizer.
- $x_k$: The $k$th initial point to find the $k$th local minimizer.
- $f(x_k)$: The function value of the $k$th initial point.
- $f(x_k^*)$: The function value of the $k$th local minimizer.

6. Conclusions

In this paper, we extend the filled function method for smooth global optimization to the case which the objective function is non-smooth, and propose an one-parameter filled function. Since the proposed filled function contains only one parameter, it can be readily adjusted at each iteration in the algorithm. We also make a numerical test. The preliminary computational results verify that the proposed filled function algorithm is promising. Because most of practical global problems have general constraints, one of our future work is to further improve this filled function method and generalize it to solve the non-smooth global problem with general constraints.

Acknowledgement
This paper was partially supported by the NNSF of China under Grant Nos.10971053 and 11001248, and by the SEDF under Grant No. 12YZ178.

7. References


Appendix:

Table 1. computational results of Problem 1

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Table 3. computational results of Problem 3

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Table 4. computational results of Problem 4

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