Cobweb - Iterative Method for Solving Nonlinear Equations

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Abstract

When we use fixed point iterative methods to solve nonlinear equations, if iterative expression construct is inappropriate, there will be not convergence of the iteration. To solve this problem, this paper introduces the cobweb model and improved the iterative algorithm, resulting in a cobweb-iterative method for solving nonlinear equations. The effectiveness and practicality of this method is further testified by numerical experiments.

Keywords: Nonlinear Equation, Cobweb Model, Symmetry, Iterative

1. Introduction

The cobweb theory [1] (cobweb theorem), is also known as the cobweb model. It use dynamic analysis of elastic theory to examine the impact of price fluctuations on the output of the next cycle, and it is a kind of theoretical model which applied to the analysis of market equilibrium.

Cobweb Model is more concentrated in the practical applications of traditional cobweb model, such as, Fenggang Yang and Lixin Cui[2] start from the economic significance of the cobweb model to clarify the application of price which used cobweb model of differential equation; Haitao Yao[3] consider the limitations of assumptions of the traditional cobweb model, using producers’ considering of frontal two period of price to decide their own period of production conditions. It builds a cobweb model which is second order linear non-homogeneous differential equation and also get stability conditions of the equilibrium price.

The numerical solution of the nonlinear equation is mainly Newton iterative method and the improved method [4].These methods are more complex, and sometimes divergent, resulting in not solving[5].

This paper, from a new angle, puts the cobweb-iteration method to solve the nonlinear equations. We can transform the solution of nonlinear equation into the point of intersection of two function \( g(x) \) and \( h(x) \). But this method requires inverse function of \( g(x) \) easy to solve. Solving process is made \( g(x_{i+1}) = h(x_i) \), and use inverse function of \( g(x) \) draw to \( x_{i+1} = g^{-1}(h(x)) \), then make a process of gradually iterative approximation intersectional; Relying on the stability of cobweb model[6], we treated the situation which closed and divergent in the iterative properly, and formed a solving nonlinear equation method which is combination of cobweb and iterative[7].

2. The method of cobweb - iteration

![Figure 1. The Solution and Intersection Point](image-url)
Write the nonlinear equations \( f(x) = 0 \) to the form of equivalence
\[
g(x) = h(x)
\]
If requires \( x^* \) satisfied \( f(x^*) = 0 \), then \( g(x^*) = h(x^*) \). Solving \( f(x) = 0 \) is to solve the point of intersection \( g(x) \) and \( h(x) \) (figure 1).

2.1. Structure equivalent form of function

Due to the solve process of spider web model needs to use a solution of inverse function of \( g(x) \) and \( h(x) \). So the structure of function \( g(x) \) and \( h(x) \) should at least ensure that there is easy an elementary function which used to ask its inverse function. For example, if we can directly make function \( g(x) \) equal to the highest order term of equation \( f(x) = 0 \), so inverse function \( g(x) \) is radical function.

2.2. Using geometry image to display the solving cobweb model

The problem for the root of equation \( g(x) = h(x) \) on the plane \( xoy \) is sure that the point of intersection of function curve \( g(x) \) and \( h(x) \) (Figure 2).

2.3. Model description

(1) Given accuracy \( \varepsilon \) and \( \delta \), select \( g(x) \) as an easy derivation function;
(2) If \( g(a) = h(a) \) (or \( g(b) = h(b) \) ), so the solution of equation \( f(x) = 0 \) is \( a \) (or \( b \) ), then,
\[
f(a) = 0 \quad \text{(or } f(b) = 0 \text{)}; \quad \text{If } g(a) \neq h(a) \text{ (or } g(b) \neq h(b) \text{)}, then turn (3);
(3) In a given interval \([a,b]\), if \( g(b) > h(b) \), we make \( x_0 = b \). Otherwise make \( x_0 = a \);
(4) Solve the solution of equation \( g^{-1}(x) = h(x_0) \) is \( x_1 \).
If \( |x_1 - x_0| > |b-a| \), then turn (5). Otherwise, continue to solve:
The solution of equation \( g^{-1}(x) = h(x_1) \) is \( x_2 \);
The solution of equation \( g^{-1}(x) = h(x_2) \) is \( x_3 \);
......
Turn (6);
(5) If \( |x_6 - x_5| > |b-a| \), namely the cobweb model is a divergent situation, then make function \( g(x) \) and \( h(x) \) about symmetric function \( g'(x) \) and \( h'(x) \), turn (4).
(6) Until $|x_n - x_{n+1}| < \varepsilon$ (including $n = 1, 2, 3, \ldots$), and $|g(x_n) - h(x_n)| < \delta$ (that is $f(x_n) < \delta$), so the solution of equation $f(x) = 0$ is $x^* = \frac{x_{n-1} + x_n}{2}$. Otherwise, the equation is without solution.

(7) If exist the closed circulation situation below figure 3, we make $x_{n+1} = \frac{x_{n-1} + x_n}{2}$, turn (4), until (6) appears.

**Figure 3. Closed Cycle Situation**

### 3. Cobweb - iteration method analysis

**Theorem 1** Any a nonlinear equation $f(x) = 0$ can be expressed as an equal form between elementary function $g(x)$ which is easy for inverse function and nonlinear function $h(x)$ [8], namely

$$f(x) = 0 \iff g(x) = h(x)$$

Proof: Set $g(x) = x$ , and $h(x) = f(x) + x$ , then, theorem 1 was established.

**Figure 4. Conversion between convergence and divergence**

**Theorem 2** If both $g(x)$ and $h(x)$ have inverse function, and the formation of the cobweb model is divergent, then it will break $f(x)$ into $-g(x)$ and $-h(x)$ , namely

$$f(x) \iff -g(x) = -h(x)$$

Then the formation of the cobweb model $-g(x)$ and $-h(x)$ is convergence (as shown in (figure 4)[9].
4. Numerical experiments

EG 1: Please solve the root of equation \( x^2 + x(\sin x)^2 - 3 = 0 \) where it is in the interval \([1, 2]\), and \( \varepsilon = 0.05 \), \( \delta = 0.1 \).

Answer: according to the cobweb model, first let equation \( x^2 + x(\sin x)^2 - 3 = 0 \) express as equivalent form \(-x^2 + 3 = x(\sin x)^2\), make \( g(x) = -x^2 + 3 \), \( h(x) = x(\sin x)^2 \). Using matlab to do the image of function \( f(x) = x^2 + x(\sin x)^2 - 3 \), \( g(x) = -x^2 + 3 \), \( h(x) = x(\sin x)^2 \) respectively, and the program is as follows:

```matlab
x=0:0.0001:2;
f=x.^2+x.*sin(x).*sin(x)-3;
g=-x.*x+3;
h=x.*sin(x).*sin(x);
plot(x,f,'*')
hold on
plot(x,g)
hold on
plot(x,h)
```

From the generated figure 5, it is known that the root of the function \( f(x) = 0 \) is the point of intersection between \( g(x) \) and \( h(x) \).

![Figure 5. Generating diagram of matlab](image)

From figure 6, if \( g(1) > h(1) \), then \( x_0 = 1 \).

The inverse function \( g(x) \) is \( x_{i+1} = \sqrt[3]{3 - g(x_i)} \), the calculation procedure with matlab as is shown in Table 1:

<table>
<thead>
<tr>
<th>Cobweb - Iterative Method for Solving Nonlinear Equations</th>
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<tbody>
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</tbody>
</table>

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Table 1. The calculation procedure of cobweb - iterative

<table>
<thead>
<tr>
<th>( h(1) = 0.7081 )</th>
<th>( \Delta x )</th>
<th>( \Delta y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(1) = 0.7081 = g(1.5139) )</td>
<td>0.5139</td>
<td></td>
</tr>
<tr>
<td>( h(1.5139) = 1.5090 = g(1.2211) )</td>
<td>0.2928</td>
<td>0.8009</td>
</tr>
<tr>
<td>( h(1.2211) = 1.0778 = g(1.3864) )</td>
<td>0.1653</td>
<td>0.4312</td>
</tr>
<tr>
<td>( h(1.3864) = 1.3398 = g(1.2885) )</td>
<td>0.0979</td>
<td>0.2620</td>
</tr>
<tr>
<td>( h(1.2885) = 1.1885 = g(1.3459) )</td>
<td>0.2928</td>
<td>0.1513</td>
</tr>
<tr>
<td>( h(1.3459) = 1.2790 = g(1.3119) )</td>
<td>0.0340</td>
<td>0.0905</td>
</tr>
</tbody>
</table>

At this time, \( |\Delta x| < \epsilon \) and \( |\Delta y| < \delta \) satisfy the iterative stop conditions, so the equations for the solution is

\[
x^* = \frac{1.3459 + 1.3119}{2} = 1.3289.
\]

EG 2: Please solve the root of equation \( e^{-0.5x} \cos(4 \pi x) = 0 \) where it is in the interval \([0, 1] \), and \( \epsilon = 0.01 \), \( \delta = 0.05 \).

Answer: according to the cobweb model, we could let equation \( e^{-0.5x} \cos(4 \pi x) = 0 \) express as equivalent form \( e^{-0.5x} \cos(4 \pi x) + x = x \), make function \( g(x) = x \), \( h(x) = x + e^{-0.5x} \cos(4 \pi x) \). Coordinates of intersections of function \( h(x) \) and function \( g(x) \) is the root of equation \( e^{-0.5x} \cos(4 \pi x) = 0 \). Using matlab to draw the geometrical relationship between the roots of the equation and Coordinates of intersections of function \( h(x) \) and function \( g(x) \) (as shown in figure 7).

The corresponding program for figure 7:

```matlab
x=0:1/100:1;
y=2*exp(-0.5*x).*cos(pi*x)+x;
plot(x,y)
hold on
z=x;
plot(x,z)
hold on
f=2*exp(-0.5*x).*cos(pi*x);
plot(x,f,'*')
```

Figure 7. Relationship of intersection point and the root
You can see from figure 7, the root of the equation $e^{0.5x} \cos(4\pi x) = 0$ is about around the place of $x = 0.5$, choice $x_0 = 0.2$ as the initial point, and function $g(x) = x$ have inverse function. Use the cobweb model to solve the root of nonlinear equation, then appear divergent iterative as shown in figure 8:

![Figure 8. Divergent iterative](image)

Under the condition of constant iterative direction, using Theorem 2, equation $f(x)$ could be decomposed into function $-g(x)$ and $-h(x)$ which is about rotational symmetry with the original decomposition function. That is $-g(x) = x$ and $-h(x) = x + e^{0.5x} \cos(4\pi x)$, As shown in figure 9. Generating symmetric function is convergence, and is clockwise, the function as easy for inverse function.

![Figure 9. Transform into convergence iteration](image)

From the above interpretation, $x_0 = 0.2$ is chosen as initial point, and function $h(x)$ is easy for inverse function, That is $x_n$ is obtained by using the function $h(x)$. In the calculation by using matlab, it is:

$x_0 = 0.2$, $-g(x_0) = -0.2000$ ;

$x_1 = \text{solve}(2*\exp(-0.5*x)*\cos(pi*x)-0.2000+x',x')=0.5823$, $-h(x_1) = -0.5823$ ;

$x_2 = \text{solve}(2*\exp(-0.5*x)*\cos(pi*x)-0.5823+x',x')=0.4791$, $-h(x_2) = -0.4791$ ;

$x_3 = \text{solve}(2*\exp(-0.5*x)*\cos(pi*x)-0.4791+x',x')=0.5054$, $-h(x_3) = -0.5054$ ;

$x_4 = \text{solve}(2*\exp(-0.5*x)*\cos(pi*x)-0.5054+x',x')=0.4986$, $-h(x_4) = -0.4986$ .

The positions of points in the rectangular coordinate system are as follows:
The corresponding matlab program for figure 10:

```matlab
x=0:1/100:1;
y=-2*exp(-0.5*x).*cos(pi*x)-x;
plot(x,y)
hold on
z=-x;
plot(x,z)
```

At the same time, make sure to Check whether \( |x_n - x_{n-1}| \) and \( |g(x_n) - h(x_n)| \) meet the requirements:

\[
|x_n - x_{n-1}| < \varepsilon \quad (n = 1, 2, 3, 4) , \quad \text{and} \quad |g(x_n) - h(x_n)| < \delta
\]

The judgment of iteration precision as is shown in Table 2:

<table>
<thead>
<tr>
<th>( -g(0.2000) = -0.2000 )</th>
<th>( \Delta x )</th>
<th>( \Delta y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(0.2000) = -0.2000 = h(0.5823) )</td>
<td>( 0.3823 )</td>
<td>( 0.3823 )</td>
</tr>
<tr>
<td>( g(0.5823) = -0.5823 = h(0.4791) )</td>
<td>( 0.1032 )</td>
<td>( 0.3823 )</td>
</tr>
<tr>
<td>( g(0.4791) = -0.4791 = h(0.5054) )</td>
<td>( 0.0263 )</td>
<td>( 0.1032 )</td>
</tr>
<tr>
<td>( g(0.5054) = -0.5054 = h(0.4986) )</td>
<td>( 0.0068 )</td>
<td>( 0.0263 )</td>
</tr>
</tbody>
</table>

When \( n = 4 \), it meets \( |x_n - x_{n-1}| < 0.01 \), and \( |g(x_n) - h(x_n)| < 0.05 \). So, Stopping iteration, the root in the interval \([0, 1]\) of equation \( e^{0.5x} \cos(4\pi x) = 0 \) is \( x = \frac{x_3 + x_4}{2} \approx 0.5020 \).

**5. Conclusions**

In this paper, the method of solving the root of algebraic nonlinear equation is to change nonlinear equation \( f(x) = 0 \) into two equal function \( h(x) \) and \( g(x) \), and it requires at least a function easy for inverse function. Namely, the iteration method for solving the intersectional of two functions; and using the stability of the cobweb model, solved the divergence appeared in the process of iteration. The theorem 1 can prove that fixed point iteration is a special circumstance of cobweb-iteration, and the divergent of fixed point iteration can become convergence through the iteration method of variable of this paper.
6. References