Output Feedback Stabilization for Discrete Nonlinear Systems Based on T-S Fuzzy Model

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Abstract

The output feedback stabilization for discrete nonlinear systems is studied based on T-S fuzzy model in this paper. Firstly, a concept of the efficient mutual non-neighboring rule group (EMNNRG) is presented. Secondly, by using the scheme of parallel distributed compensation (PDC) and the piecewise fuzzy Lyapunov function, the stability of the closed-loop discrete T-S fuzzy control systems is analyzed and a corresponding fuzzy controller is designed, and the quadratic stability condition of the closed-loop discrete T-S fuzzy systems only requires meeting the stability condition of the fuzzy Lyapunov function approach in each EMNNRG. Finally, simulation study for an illustrative example is conducted to show the effectiveness of the proposed method.

Keywords: Feedback Stabilization, Nonlinear Systems, T-S Fuzzy Model

1. Introduction

During the past years fuzzy control has emerged as one of the most active and fruitful areas\cite{1}. However, one of the weak points of the fuzzy controller design is lacking the efficient methods of stability analysis and systematic design for fuzzy control systems. Stability analysis of fuzzy control systems has been difficult because fuzzy control systems are essentially nonlinear systems\cite{2}. Recently, many stability methods\cite{3-5} have been proposed to check the stability of the discrete T-S fuzzy model by which nonlinear systems can be identified. Reference [5] addressed the problem of designing robust static output-feedback controllers for nonlinear discrete interval systems with time delays both in states and in control input. In the approach, sufficient conditions for guaranteeing the robust stability for the considered systems were derived in terms of the matrix spectral norm of the closed-loop fuzzy system. The stability conditions were further formulated into linear matrix inequalities so that the desired controller can be easily obtained by using the Matlab linear matrix inequality (LMI) toolbox. Reference [6] proposed a new quadratic stabilization condition for T-S fuzzy control systems. Based on the LMI-based conditions derived, the controllers for stabilizing T-S fuzzy control systems can be easily synthesized. In comparison with conventional conditions, the proposed condition was not only suitable for designing fuzzy state feedback controllers but also convenient for fuzzy static output feedback controller design. Since only a set of LMIs is involved, the controller design is quite simple and numerically tractable. In Reference [7] the output stabilization problem of T-S fuzzy models was considered. First a natural form of observers for such models was given. Sufficient conditions for their asymptotic convergence were given which were dual to those for the stability of state feedback fuzzy controllers. Then by designing a state feedback controller and an observer the stabilizing property of the control and the asymptotic convergence of the observer were guaranteed by the Lyapunov method using positive definite matrices. Reference [8] studied the stabilization and robustness problem for a fuzzy time-delay system. Based on Lyapunov criterion and Razumikhin theorem, some sufficient conditions were derived under which the state feedback can stabilize the whole fuzzy time-delay system asymptotically. LMIs were utilized to solve the problem role in this paper. In Reference [9], an approach to design a static output-feedback control for discrete-time-delay fuzzy models was proposed. Based on a quadratic Lyapunov function, sufficient conditions in terms of bilinear matrix inequalities were derived for asymptotic stability. To design a static output-feedback stabilizing controller, sufficient conditions were formulated into a set of LMIs by using some matrix transformations, which made possible to calculate directly the desired controller.

However Reference [5-9] designed the fuzzy controller based on a single quadratic Lyapunov function, and finding a common positive definite matrix is difficult when the number of rules is large in the fuzzy control systems. Therefore, Reference [10-14] proposed the design methods by using the
fuzzy Lyapunov function. Reference [10] addressed stability analysis and stabilization for T-S fuzzy systems via a so-called fuzzy Lyapunov function which is a multiple Lyapunov function. The fuzzy Lyapunov function was defined by fuzzily blending quadratic Lyapunov functions. Based on the fuzzy Lyapunov function approach, stability conditions for open-loop fuzzy systems and stabilization conditions for closed-loop fuzzy systems were given. Reference [11] dealt with T-S systems stabilization based on dynamic output feedback compensators (DOFC). For T-S uncertain closed-loop systems non-quadratic stability conditions were proposed. Based on a fuzzy Lyapunov candidate function and the descriptor redundancy property, these stability conditions were written in terms of LMIs. Afterward, the DOFC was designed with $H_\infty$ criterion in order to minimize the influence of external disturbances. In Reference [12], new robust $H_\infty$ controller design methodologies for T-S descriptors were considered. Based on LMIs, two different approaches were proposed. The first one involved a “classical closed-loop dynamics” formulation and the second one a “redundancy closed-loop dynamics” approach. The provided conditions were obtained through a fuzzy Lyapunov function candidate and a non-PDC control law. Both the classical and redundancy approaches were compared. It was shown that the latter led to less conservative stability conditions. Reference [13] investigated the problems of stability analysis and stabilization for a class of discrete T-S fuzzy system with time-varying state delay. Based on a novel fuzzy Lyapunov-Krasovskii function, a delay partitioning method was developed for the delay-dependent stability analysis of fuzzy time-varying state delay systems. As a result of the novel idea of delay partitioning, the proposed stability condition was much less conservative than most of the existing results. A delay-dependent stabilization approach based on a nonparallel distributed compensation scheme was given for the closed-loop fuzzy systems. Reference [14] proposed a new Lyapunov–Krasovskii function to cope with stability analysis and control design for time-delay nonlinear systems modeled in the T-S fuzzy form. By using the Gu’s discretization technique and by employing an appropriated fuzzy function, less conservative conditions were obtained.

The stability analysis methods proposed by Reference [10-14] need solving more LMIs, so there is a room to reduce the number of LMIs solved for the controller design of T-S fuzzy system. In this paper, the concept of the efficient mutual non-neighboring rule group (EMNNRG) is defined firstly. Then, based on the piecewise fuzzy Lyapunov function, a relaxed stability condition of the closed-loop discrete T-S fuzzy systems is analyzed and a fuzzy controller is designed. The feasibility of the proposed method is illustrated by a simulation example.

The paper is organized as follows. Section 2 presents an overview of discrete T-S fuzzy control systems and recalls an existing stability condition. In Section 3, EMNNRG is defined, and based on PDC method, the stability analysis and controller design for the closed-loop discrete T-S fuzzy control system are presented. In Section 4, the system simulation is conducted to show the effectiveness of this method for designing a discrete T-S fuzzy control system. Finally, concluding remarks are given in Section 5.

2. Discrete T-S fuzzy system and basic stability condition

A nonlinear system can be represented as follows:

$$R_i: \text{IF } x_i(k) \text{ is } M_{i}^{j} \text{ and } \cdots \text{ and } x_n(k) \text{ is } M_{n}^{j} \text{ THEN } x(k+1) = A_i x(k) + B_i u(k)$$

(1)

where $i = 1, 2, \cdots, r$, $r$ and $n$ are the numbers of rules and input variables respectively, $x(k) = [x_1(k), x_2(k), \cdots, x_n(k)]$ is the state vector, $u(k)$ is the control input, $M_j^j$ ($j = 1, \cdots, n$) is the fuzzy set. By the singleton fuzzifier, the product inference engine and center average defuzzification, the final output of (1) is inferred as:

$$x(k+1) = \sum_{j = 1}^{r} h_i(k)(A_i x(k) + B_i u(k))$$

(2)

where $h_i(k) = \prod_{j = 1}^{n} M_j^j(x_j(k)) / \sum_{j = 1}^{r} \prod_{j = 1}^{n} M_j^j(x_j(k))$, and $\sum_{j = 1}^{r} h_i(k) = 1$. 


A sufficient condition to check the stability of the discrete control systems based on Lyapunov direct method can be expressed as:

**Lemma 1.** Consider the discrete-time systems described by \( x(k+1) = f(x(k)) \), where \( x(k) \in \mathbb{R}^n \), \( f(x(k)) \) is a \( n \times 1 \) function vector and satisfies \( f(0) = 0 \). If there exists a continuous scalar function \( V(x(k)) \) satisfying

1. \( V(0) = 0 \),
2. \( V(x(k)) > 0 \) for \( x(k) \neq 0 \),
3. \( V(x(k)) \to \infty \) as \( |x(k)| \to \infty \),
4. \( L = V(x(k+1)) - V(x(k)) < 0 \), for \( x(k) \neq 0 \),

then the equilibrium state \( x(k) = 0 \) is asymptotically stable in the large, and \( V(x(k)) \) is a Lyapunov function.

A controller design procedure can be presented for the fuzzy system given in (2) based on the parallel distributed compensation (PDC) approach. For (1), let \( K_i \) denote the state feedback gain of the \( i \)th local model. From this, the local control laws are as follows:

\[
R_i : \text{IF } x_i(k) \in M_i^L \text{ and } \cdots \text{ and } x_n(k) \in M_n^L \text{ THEN } u(k) = -K_i x(k), \quad i = 1, 2, \cdots, r.
\]

The global model of a fuzzy controller can be inferred as follows:

\[
u(k) = - \sum_{i=1}^{r} h_i(k) K_i x(k)
\]

### 3. Controller design of the closed-loop discrete T-S fuzzy systems

In this paper, all of discussions and results are aimed at the prescribed fuzzy partitions which are complete and concise as Reference [15], and shown in Fig. 1. It is assumed that the total number of fuzzy partitions of each premise variables is even, i.e., \( q_j \in \{2, 4, \cdots, 2k\} \), \( k \in \mathbb{Z}^+ \), \( q_j \) denotes the number of the fuzzy partitions of the \( j \)th premise variable. \( j = 1, 2, \cdots, n \), and \( n \) denotes the number of premise variables.

**Definition 1.** The fuzzy partitions of each premise variables are divided into \( \sum_{m=2}^{q_j} F_j^{m-1} \) and \( \sum_{m=1}^{q_j} F_j^{m+1} \). Therefore, \( \left( F_j^{m+1}, F_j^{m+1} \right) \) is said to be an overlapped fuzzy partition pair, \( m \in [1, 2, \cdots, (q_j - 1)] \). The operating region structured by \( n \) overlapped fuzzy partition pairs is said to be a rule group. For \( \forall m \in [1, 3, \cdots, (q_j - 1)] \), \( \left( F_j^{m}, F_j^{m} \right) \) is said to be an efficient mutual non-neighboring overlapped fuzzy partition pair. The operating region structured by \( n \) efficient mutual non-neighboring overlapped fuzzy partition pairs is said to be an efficient mutual non-neighboring rule group (EMNNRG).
Proposition 1. The number of EMNNRGs is \( \prod_{j=1}^{q} \left( \frac{q_j}{2} \right) \).

Proposition 2. The rules included in a random rule group are all included in EMNNRGs.

Definition 2. For a fuzzy system described by (2) with SFP inputs, if any of overlapped-rules groups is described as \( g_c \), \( c=1,2,\cdots,f \), a discrete piecewise fuzzy Lyapunov function is defined as

\[
V(x(k)) = x^T(k)P(k)x(k), \quad P(k) = \sum_{c=1}^{f} \lambda_c P_c(k),
\]

where \( \lambda_c(x(k)) = \begin{cases} 1 & x(k) \in g_c, \\ 0 & x(k) \notin g_c \end{cases} \), \( \sum_{c=1}^{f} \lambda_c(x(k)) = 1 \), \( P_c(k) = \sum_{h_c} h_c(k)P_c \), \( f \) denotes the number of overlapped-rules groups, and \( L_c = \{ \text{the sequence numbers of rules included in } g_c \} \).

Lemma 2. Assume that \( P, P_l, P \) are the positive definite matrices, if there exist matrices \( A \in \mathbb{R}^{nxn} \) and \( B \in \mathbb{R}^{nxm} \) which satisfy \( A^TPA - P < 0 \) and \( B^TPB - P < 0 \), then the following inequality comes into existence:

\[
A^TPB - P + B^TPA - P < 0
\]

Proof. For a random real number vector \( x \in \mathbb{R}^n \) and \( x \neq 0 \), we have

\[
\Gamma = x^T \left( A^TPA - P + B^TPB - P \right)x = x^T \left[ - (A - B)^T \right] \left( A^TPA - P + B^TPB - P \right)x + x^T \left( A^TPA - P \right)x + x^T \left( B^TPB - P \right)x.
\]

Due to \( (A - B)x \) \( \geq 0 \), \( x^T (A^TPA - P)x < 0 \) and \( x^T (B^TPB - P)x < 0 \), there is \( \Gamma < 0 \). Therefore, \( A^TPB - P + B^TPA - P < 0 \) comes into existence. Q.E.D.

Theorem 1. For a fuzzy control system described by (2) and (4), let \( G_{k} = A - B_K \), then the equilibrium of the closed-loop fuzzy control systems is asymptotically stable in the large if there exist positive definite matrices \( P_l \) or \( P_c \) in each EMNNRG such that

\[
G_{k}^T P G_{k} - P < 0, \quad i,k,l \in \{ \text{the sequence numbers of rules included in } G_c \},
\]

where \( G_c \) denotes the \( g \)th EMNNRG, \( g = \prod_{j=1}^{q} \left( \frac{q_j}{2} \right) \) and \( q_j \) denotes the number of fuzzy partition of the \( j \)th input variable.

Proof. If \( x(k) \) and \( x(k+1) \) are in the same overlapped-rules group, and \( h_j(k)h_i(k) \geq 0 \), \( \sum_{i_L} \sum_{k_L} h_i(k)h_i(k) = 1 \), \( \sum_{i_L} h_j(k) + 2 \sum_{i_L} h_j(k)h_i(k) = 1 \), then local model in the \( c \)th overlapped-rules group can be expressed as:

\[
x(k+1) = \sum_{i_L} h_i(k) \left( A_x(k) - B_i \left( \sum_{i_L} h_i(k)K_i, x(k) \right) \right) = \sum_{i_L} \sum_{k_L} h_i(k)h_i(k) \left( A - B_K \right) x(k).
\]

If there exist positive definite matrices \( P_l \) or \( P_c \) satisfying (7), then \( V_c(x(k)) = x^T(k) \sum_{i_L} h_i(k)P_i x(k) \) is chosen as a Lyapunov function in the \( c \)th overlapped-rules group, in which it can be easily validated that \( V_c(x(k)) \) satisfies (a), (b) and (c) of Lemma 1.
From (4), (7) and with $h_i(k) > 0$, for $\forall x(k) \neq 0$, we have

$$\Delta V_i(x(k)) = V_i(x(k+1)) - V_i(x(k))$$

$$= \left[ \sum_{i \in I_k} \sum_{k \in U} h_i(k)G_{\alpha, i}x(k) \right] \left[ \sum_{i \in I_k} \sum_{k \in U} h_i(k)G_{\alpha, i}x(k) \right] - x^T(k)\sum_{i \in I_k} h_i(k)P_i x(k)$$

$$= x^T(k) \sum_{i \in I_k} h_i(k+1)A_i x(k)$$

$$= x^T(k) \left( \sum_{i \in I_k} h_i(k+1)A_i \right) x(k).$$

where

$$A = \sum_{i \in I_k} \sum_{k \in U} \left( h_i(k)h_i(k) \hat{h}_i(k)(G_{\alpha, i}P_iG_{\alpha, i} - P_i) \right)$$

$$= \sum_{i \in I_k} \sum_{k \in U} h_i^2(k)h_i(k)(G_{\alpha, i}P_iG_{\alpha, i} - P_i) + \sum_{i \in I_k} \sum_{k \in U} h_i^2(k)h_i(k)(G_{\alpha, i}^TP_iG_{\alpha, i} - P_i) + \sum_{i \in I_k} \sum_{k \in U} h_i^2(k)h_i(k)(G_{\alpha, i}^TP_iG_{\alpha, i} - P_i).$$

From (6) we have $(G_{\alpha, i}^TP_iG_{\alpha, i} - P_i) < 0$ and $(G_{\alpha, i}P_iG_{\alpha, i} - P_i) < 0$, which result in $A < \sum_{i \in I_k} \sum_{k \in U} h_i^2(k)h_i(k)(G_{\alpha, i}P_iG_{\alpha, i} - P_i).$

So, $\Delta V_i(x(k)) < x^T(k) \sum_{i \in I_k} \sum_{k \in U} h_i^2(k)h_i(k)(G_{\alpha, i}^TP_iG_{\alpha, i} - P_i)x(k).$ Because of $G_{\alpha, i}^TP_iG_{\alpha, i} - P_i < 0$, then $\Delta V_i(x(k)) < 0$ would be guaranteed. Therefore, $V_i(x(k))$ also satisfies (d) of Lemma 1 in the cth overlapped-rules group.

If $x(k)$ and $x(k+1)$ are not in the same overlapped-rules group, on the other hand, then the global model of the fuzzy system in the whole universe can be expressed as:

$$x(k+1) = \sum_{i \in I_k} \hat{A}_{\alpha, i} \left( x(k) \right) \sum_{i \in I_k} \hat{h}_i(k)h_i(k)G_{\alpha, i}x(k).$$

Using (5) we have

$$V_i(x(k)) = x^T(k)P_i x(k) = x^T(k) \left( \sum_{i \in I_k} \hat{A}_{\alpha, i} \hat{P}_i \right) x(k) = \sum_{i \in I_k} \hat{A}_{\alpha, i} x^T(k)P_i x(k) = \sum_{i \in I_k} \hat{A}_{\alpha, i} V_i(x(k)).$$

It can be easily validated that $V_i(x(k))$ satisfies (a), (b) and (c) of Lemma 1. For $\forall x(k) \neq 0$, we have

$$\Delta V_i(x(k)) = V_i(x(k+1)) - V_i(x(k)) = \sum_{i \in I_k} \hat{A}_{\alpha, i} V_i(x(k+1)) - \sum_{i \in I_k} \hat{A}_{\alpha, i} V_i(x(k))$$

$$= \sum_{i \in I_k} \hat{A}_{\alpha, i} \left( V_i(x(k+1)) - V_i(x(k)) \right) = \sum_{i \in I_k} \hat{A}_{\alpha, i} \Delta V_i < 0,$$

then $V_i(x(k))$ also satisfies (d) of Lemma 1.

From Proposition 2, the rules included in a random rule group are all included in EMNNRGs. Therefore, we only need finding positive definite matrix $P_i$ (or $P_i'$) to satisfy (7) in each EMNNRG, then the equilibrium of the fuzzy system described by (2) is asymptotically stable in the large. Q.E.D.

4. Simulation analysis

In this section, a discrete fuzzy system is considered as follows:

$$R_i: \text{IF } x_i(k) \text{ is } M_{\alpha, i}^1 \text{ and } x_{i,1}(k) \text{ is } M_{\alpha, i}^1 \text{ THEN } x(k+1) = A_i x(k) + B_i u$$

(8)
where $i = 1, 2, \ldots, 36$. The fuzzy partitions of $x_1(k)$ and $x_2(k)$ are shown in Fig. 1.

The fuzzy sets and the system parameters are shown as follows:

\[
M_1^1 = M_2^1 = M_3^2 = M_4^3 = F_1^1, \quad M_5^1 = M_6^1 = M_7^2 = M_8^3 = F_1^2, \\
M_9^1 = M_1^4 = M_2^4 = M_3^5 = M_4^6 = F_1^3, \quad M_5^4 = M_6^4 = M_7^5 = M_8^6 = F_2^1, \\
M_1^7 = M_2^7 = M_3^8 = M_4^9 = F_1^4, \quad M_5^7 = M_6^7 = M_7^8 = M_8^9 = F_2^2, \\
M_1^3 = M_2^3 = M_3^4 = M_4^5 = M_5^6 = M_6^7 = M_7^8 = M_8^9 = F_2^3.
\]

\[
A_1 = A_{10} = A_{19} = A_{28} = \begin{bmatrix} -0.43 & 2 \\ 0 & -7.16 \end{bmatrix}, \quad A_1 = A_{11} = A_{20} = A_{29} = \begin{bmatrix} -5 & 2 \\ 2 & -6 \end{bmatrix}.
\]

![Fig. 2. The sketch map of fuzzy partitions](image-url)
The closed-loop eigenvalues of the 36 local linear subsystems via state feedback are selected as
$$\begin{bmatrix} 1 & 2 & \cdots & 36 \end{bmatrix}$$
$$\begin{bmatrix} 0.5 & 0.75 \end{bmatrix} \cdots$$
P. The state feedback gain of the local linear subsystems can be derived from Ackermann’s formula as follows:
$$\begin{bmatrix} 11 & 01 & 9 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 0.6537 & 0.8512 \end{bmatrix}$$
$$\begin{bmatrix} 21 & 12 & 02 & 9 & 00 \end{bmatrix} = \begin{bmatrix} 0.7769 & 0.5432 \end{bmatrix}$$
$$\begin{bmatrix} 31 & 2 & 2 & 13 & 0 \end{bmatrix} = \begin{bmatrix} 0.0863 & 0.1734 \end{bmatrix}$$
$$\begin{bmatrix} 41 & 32 & 23 & 1 \end{bmatrix} = \begin{bmatrix} 0.7689 & 0.6553 \end{bmatrix}$$
$$\begin{bmatrix} 51 & 4 & 2 & 33 & 2 \end{bmatrix} = \begin{bmatrix} 0.4437 & 0.5648 \end{bmatrix}$$
$$\begin{bmatrix} 61 & 52 & 43 & 3 \end{bmatrix} = \begin{bmatrix} 0.2475 & 0.1263 \end{bmatrix}$$
$$\begin{bmatrix} 71 & 62 & 53 & 4 \end{bmatrix} = \begin{bmatrix} 0.477 & 0.2558 \end{bmatrix}$$
$$\begin{bmatrix} 81 & 7 & 2 & 63 & 5 \end{bmatrix} = \begin{bmatrix} 0.5226 & 0.1774 \end{bmatrix}$$
$$\begin{bmatrix} 91 & 82 & 73 & 6 \end{bmatrix} = \begin{bmatrix} 0.6637 & 0.1389 \end{bmatrix}$$

There are 9 EMNNRGs (\(G_1, G_2, G_3, G_11, G_13, G_17, G_23, G_25\)) in the fuzzy system (8). By using Theorem 1 36 positive definite matrices are found in order to check the stability of the fuzzy system (8).

The fuzzy system (8) is simulated under various initial conditions. The simulation results show that this system is stable under all initial conditions. The system state responses under the initial condition of \(x_k(0) = [1 \ 1]^T\) are shown in Fig. 3.
5. Conclusion

A controller for discrete T-S fuzzy systems has been designed in terms of the definition of EMNNRG and the piecewise fuzzy Lyapunov function. This stability condition only requires meeting the stability condition of the fuzzy Lyapunov function approach in each EMNNRG. To demonstrate the effectiveness of the proposed controller design method a simulation example has been considered. Simulation result has verified the validity of the proposed design method.

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7. References