Tensor Approximation Algorithm based on High Order Singular Value Decomposition

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Abstract

To solve some problems which JPEG compression obtains results of poor reconstruction quality and high computational complexity for image containing more high frequency information, a novel tensor approximation algorithm based on high order singular value decomposition has been proposed. The new algorithm respects each image both gray image and color image as a high order tensor. It transforms the image into singular value matrix that contains nonzero singular values to implement image compress and discards singular sub-tensor corresponding smaller singular values in tensor decomposition to reduce the calculation work. The experiment result compared with JPEG shows that the algorithm has better performance than JPEG for color images. It is easy to apply the algorithm to any high order tensor in computer vision.

Keywords: Image Compression, Tensor Approximation Algorithm, Singular Value Decomposition, High Order Singular Value Decomposition, Jpeg Compression

1. Introduction

Image compression is a promising research field, where the technology's breakthroughs will bring out further affect on both communication and multimedia fields [1]. On the past few decades, many applications require processing a large amount of multi-dimension data. Uncompressed multimedia data requires considerable storage capacity and transmission bandwidth. Despite rapid progress in mass-storage density, processor speeds, and digital communication system performance, demand for data storage capacity and data-transmission bandwidth continues to outstrip the capabilities of available technologies. The recent growth of data intensive multimedia-based web applications have not only sustained the need for more efficient ways to encode signals and images but have made compression of such signals central to storage and communication technology.

Although an international standard for still image compression-JPEG standard has been established by ISO and IEC, the performance of such coders generally degrade at low bit-rates mainly because of the underlying block-based Discrete Cosine Transform (DCT) scheme [2]. However, with the proliferation of multimedia applications, the traditional JPEG compression technology has been unable to meet the requirements of the multimedia image data. Therefore, a new generation of static image compression technology JPEG 2000, which has higher compression ratio and more new function, was born. Wavelet transform has become a cutting edge technology for image compression research. The top contenders in the JPEG-2000 standard are all wavelet-based compression algorithms [3, 4]. However, traditional image compression technique is all vector-based method, which is harmful to preserve data correlation. Tensor method is proposed to keep intrinsic structure of data in the process of storing data, in which multi-media data can be respect as high order tensor [5-7]. In multilinear algebra, there is not a general decomposition method for data-tensors with all the properties of matrix singular value decomposition (SVD). More recently, SVD is applied as a powerful image compression technique because of its unique ability to split up data space into orthogonal signal and noise subspaces [8-10]. High-order singular value decomposition (HOSVD) is a generalization of SVD to the case of parametric spaces having more than two dimensions by L. De Lathauwer [11]. It conquers the disadvantage of SVD that is only suitable for matrix format. Historically, much of the interest in higher-order SVD was driven by the need to analyze empirical data, especial in psychometrics and chemometrics. HOSVD has been recently applied to image processing [12-13].
In the paper, we respect color image as third order tensor with row, column and color modes. Firstly, we prove that HOSVD is extension of matrix SVD, especially for second-order tensor (matrix), both is equivalent. Therefore, it’s easy to see that gray image compression based on HOSVD is the same as SVD. Then, considering the characteristics of energy concentration, we discard singular sub-tensor corresponding smaller singular values in tensor decomposition to reduce the calculation work. So we introduce tensor approximation algorithm and apply it to realize image compression.

2. JPEG 2000

As a new generation of international standards for still image compression, JPEG2000 has more advantages than the original standard in many ways [14]. It adopts some of the latest achievements of image encoding, such as discrete wavelet transform (DWT), optimization of embedded block truncation coding EBCOT and adaptive arithmetic encoder and so on. The first stage of JPEG 2000 encoding (Tier-1) includes the original image discrete wavelet transform, transform coefficients quantization, entropy coding and so on. Then the bit stream format of JPEG 2000 will be organized, called the second stage (Tier-2) of encoding. Decoding is the reverse process of encoding.

2.1. Preprocessing

Preprocessing mainly includes tiling, DC level shifting and component transformation. First of all, source image is divided into non-overlapping rectangular piece. Each piece can be encoded independently to reduce the size of required buffer, which can be executed in parallel mode. Moreover, it enables low-complexity encoding by selecting part of the image to the decoder.

2.2. Wavelet transform

Wavelet transform has the characteristics of Multi-resolution Analysis, and it can reflect the local character of signal. Discrete wavelet transformation (DWT) has become very popular as the basis of many recent image processing and compression techniques and has been adopted by new image compression standard, i.e., JPEG2000. Wavelet coefficient image obtained by discrete wavelet transform on each component of the image. Through multi-level wavelet decomposition, wavelet coefficients can not only describe high-frequency information in the local region, but also low-frequency information of the image. The purpose of wavelet transform is to remove the correlation of pixels of each sub-image, as much as possible to centralize the image information at the small part of the transform coefficients. And then the coefficients which has less information will be quantified by zero because it has little impact on sub-images of reconstruction. To reduce the computational complexity and memory, JPEG2000 standard supports the algorithm based on lifting wavelet.

2.3. Quantization

In lossy coding mode, we need the quantization of various sub-bands in order to achieve the purpose of compression. The steps of quantization are not required for lossless compression. The quantization of wavelet coefficient is one of the most important reasons to information. Although the results of rough quantify will obtain the higher compression ratio, it also reduce the quality of the reconstructed image. JPEG2000 standard uses uniform quantization. The key is to choose the right step to ensure that the decoder can recover the expected image and obtain greater compression ratio.

2.4. Core coding algorithm

Quantized wavelet coefficients need to be organized effectively after quantization. JPEG2000 used the optimal algorithms of truncated embedded block coding with optimized truncation (EBCOT) [15]. After the wavelet transform of image, which generated sub-band will be divided into small rectangular code blocks. Each code block use the approach based on bit-plane to code independent, and adopt arithmetic coding based on content to compress the model and generate the compressed bit stream. In order to achieve the target bit rate of image coding, for each code block bit-plane calculated a few split
point at the rate-distortion curve, and thus to allocate each code block for the optimal number of bits. Therefore, in coding, the first of each code block with a higher bit rate to encoding, and then calculated according to the optimal bit rate allocation to determine the split point to truncate the bit stream for each block. Corresponds the set of target bit rate, you can determine the corresponding optimal split point set. Once the code block bit stream is truncated to achieve the target bit rate, according to the usual bit-plane order to re-organize and form an embedded bit stream. The coding algorithm of EBCOT obtain the bit stream will have some characteristics with multi-resolution and multi-level quality optimize and so on. And the rate-distortion performance is better than SPIHT algorithm, but also higher complexity than the SPIHT algorithm.

3. High order singular value decomposition

An Nth-order tensor is denoted as \( A \in R^{l_1\times l_2\times \ldots\times l_N} \). It is addressed by N indices \( i_n \) n=1... N. Each \( i_n \) addresses the n-mode of \( A \) , and the number of dimensions N is called as the order (modes) of \( A \) . Especially, a scalar is zeroth-order tensor, a vector is a first-order tensor and a matrix is a second-order tensor.

Definition 1: (scalar product of tensor) the scalar product \( \langle A, B \rangle \) of two tensor \( A, B \in R^{l_1\times l_2\times \ldots\times l_N} \) is represented as \( \langle A, B \rangle = \sum_{i_1} \sum_{i_2} \ldots \sum_{i_N} a_{i_1\ldots i_N} \times b_{i_1\ldots i_N} \).

Definition 2: (Frobenius-norm) The Frobenius-norm of a tensor \( A \) is defined as \( \| A \| = \sqrt{\langle A, A \rangle} \).

Definition 3: (Tensor distance) The distance between \( A \) and \( B \) is defined by \( D(A, B) = \| A - B \| \).

Definition 4: (Mode-n Product) Given tensor \( A \in R^{l_1\times l_2\times \ldots\times l_N} \) and matrix \( U \in R^{l_i\times l_j} \), the product of \( A \) and \( U \) called the mode-n product can result in another tensor \( B \in R^{l_1\times l_2\times \ldots\times l_N} \) which can be given as: \( B = A \times U = \sum_{i_1} \sum_{i_2} \ldots \sum_{i_N} a_{i_1\ldots i_N} \times U_{i_N} \).

Definition 5 (Mode-k Unfolding): The mode-k matrix unfolding of an Nth order tensor \( A \in R^{l_1\times l_2\times \ldots\times l_N} \) can be represented by \( A_{(k)} \in R^{l_k \times \prod_{j \neq k} l_j} \), and its column index is equal to \( j = 1 + \sum_{j = k+1}^n (l_j - 1)J_k \).

Definition 6: the n-rank of \( A \), denoted by \( \text{rank}_n(A) = \text{rank}(A_{(n)}) \), is the dimension of the vector space spanned by the n-mode vectors.

Observation 1: A tensor is often rearrangement of matrix. Considering second-order tensor \( A \in R^{l_1\times l_2} \) which is naturally a matrix, its mode-1 matrix unfolding \( A_{(1)} \) is equal to itself and mode-2 matrix unfolding \( A_{(2)} \) is equal to \( A^T \). Observation 1 is helpful for us to connect tensor and its unfolding matrix and useful to proof theorem 1.

Definition 7: The Frobenius-norms \( \| S_{(n)} \| \), symbolized by \( \alpha_{(n)} \), are n-mode singular values of \( A \) and the vector \( U_{(n)}^{(n)} \) is an n-th mode singular vector.

Definition 8: Any tensor \( A \in R^{l_1\times l_2\times \ldots\times l_N} \) can be expressed as the product: \( A = B \times U^{(1)} \times U^{(2)} \times \ldots \times U^{(N)} \) With the following properties:

1. \( U^{(n)} = (U_1^{(n)} U_2^{(n)} \ldots U_l^{(n)}) \) \( \)is a unitary \( (l \times l) \) matrix;

2. The core tensors \( B_{(n)} = \alpha \) of \( B \in R^{l_1\times l_2\times \ldots\times l_N} \) have the following properties:

   All-orthogonality: two sub-tensors of the core tensor \( B_{(n)} \) and \( B_{(n)} \) are orthogonal for all possible values of \( n \), \( \alpha \) and \( \beta \) subject to when \( \alpha \neq \beta \), then \( \langle B_{(n)} , B_{(\beta)} \rangle = 0 \);

   Ordering: the sub-tensors in the core tensor \( B \) are ordered according to their Frobenius-norm,
\[ \mathbf{B}_{i} = \mathbf{B}_{i+1} \geq \mathbf{B}_{i+2} \geq \ldots \geq \mathbf{B}_{i+N} \geq 0, \text{ for } n=1, \ldots, N. \]

The matrix representation of HOSVD can be obtained by unfolding \( \mathbf{A} \) and \( \mathbf{B} \):
\[
\mathbf{A}_{(n)} = \mathbf{U}^{(n)^T} \cdot \mathbf{B}_{(n)} \cdot \left( \left( \mathbf{U}^{(n+1)} \otimes \mathbf{U}^{(n+2)} \otimes \ldots \otimes \mathbf{U}^{(N)} \otimes \mathbf{U}^{(1)} \otimes \mathbf{U}^{(2)} \otimes \mathbf{U}^{(n-1)} \right) \right)
\]  
(1)

where \( \otimes \) denotes Kronecker product.

Defining a diagonal matrix \( \mathbf{I}_{n, n} \in \mathbb{R}^{n \times n} \) and a column-wise orthonormal matrix \( \mathbf{V}^{(n)} \in \mathbb{R}^{I_{n, n} \times I_{n, n}} \) as follows:
\[ \mathbf{I}_{n, n} = \text{diag}(\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)}) \]
\[ \mathbf{V}^{(n)} = \mathbf{B}_{(n)} \cdot \left( \left( \mathbf{U}^{(n+1)} \otimes \mathbf{U}^{(n+2)} \otimes \ldots \otimes \mathbf{U}^{(N)} \otimes \mathbf{U}^{(1)} \otimes \mathbf{U}^{(2)} \otimes \mathbf{U}^{(n-1)} \right) \right) \]
where \( \mathbf{B}_{(n)} = \sum^{n}_{i=1} \mathbf{B}_{i} \).

Then it is clear that HOSVD lead to an SVD of the matrix unfolding:
\[ \mathbf{A}_{(n)} = \mathbf{U}^{(n)^T} \cdot \mathbf{V}^{(n)^T} \]  
(2)

Equation (2) indicates that HOSVD of a given tensor \( \mathbf{A} \in \mathbb{R}^{I_{1} \times I_{2} \times \ldots \times I_{N}} \) can be computed by two steps [16]:

1. The computation of projection matrices: projection matrices \( \mathbf{U}^{(n)} \) correspond to the left singular matrices of mode-\( n \) matrix unfolding \( \mathbf{A}_{(n)} \);
2. The computation of core tensor: \( \mathbf{B} = \mathbf{A} \times_{1} \mathbf{U}^{(1)^T} \times_{2} \mathbf{U}^{(2)^T} \times_{N} \mathbf{U}^{(N)^T} \).

Theorem 1: HOSVD is expansion of matrix SVD. Especially, for second-order tensor (matrix), HOSVD boils down to matrix SVD.

Proof: It is obvious that HOSVD is expansion of matrix SVD from both of definitions.

For second-order tensor \( \mathbf{A} \in \mathbb{R}^{I_{1} \times I_{2}} \), it is naturally a matrix. In the light of matrix SVD, \( \mathbf{A} \) can be represented by equation (3):
\[
\mathbf{A} = \mathbf{U} \sum \mathbf{V}^T
\]  
(3)

\( \mathbf{A} \) also can be decomposed by HOSVD:
\[
\mathbf{A} = \mathbf{B} \times_{1} \mathbf{U}^{(1)} \times_{2} \mathbf{U}^{(2)}
\]  
(4)

Equation (4) can also be reformulated with simple algebraic computation:
\[ \mathbf{A} = \mathbf{U}^{(1)^T} \mathbf{B} \mathbf{U}^{(2)^T} \]  
(5)

According analysis about computation of HOSVD above, it is easy to get that \( \mathbf{U}^{(1)} \) is the left singular matrix of \( \mathbf{A}^{(1)} \), which means \( \mathbf{U}^{(1)^T} = \mathbf{U} \).

Similarly, \( \mathbf{U}^{(2)} \) corresponds to the left singular matrix of \( \mathbf{A}^{(2)} \). From observation 1, we can get equation (6):
\[ \mathbf{A}^{(2)} = \mathbf{A}^T = \mathbf{V} \sum \mathbf{U}^T \]  
(6)

Then, it’s easy to get equation (7):
\[ U(2) = V \]  

Hence, the core tensor can be computed by equation (6) and (7):

\[ B = U^{(1)} V^{(2)} = U^T A V \]  

As the result of equation (128), it is obvious that HOSVD is equal to matrix SVD for second-order tensor. As described in theorem 1, it is easy to get that because gray image is second order tensor, HOSVD based on gray image is equal to its SVD. HOSVD is an extension of SVD which only applies to matrices to third or larger order tensors.

### 4. Tensor approximation algorithm

#### 4.1. Image compression processing based on SVD

Singular value decomposition is an orthogonal transform of non-symmetric matrix. Any matrix can be decomposed into the singular value matrix only containing several non-zero value. It is applied as a powerful image compression tools because of its ability to significantly reduce the capacity of the image while there is no loss of the recovered image [17]. Image compression processing based on SVD can be described as the following:

**Step 1** SVD of a given tensor \( A \) can be computed by the following formula:

\[
A = U S V^T = U \left[ \begin{array}{ccc}
0 & \cdots & 0
\end{array} \right] V^T = \sum_{i=1}^{r} A_i = \sum_{i=1}^{r} \sigma_i u_i v_i^T
\]  

**Step 2** Set the image compression ratio after decomposition of matrix \( A \). Select the number of singular value \( K \), and discard the extra singular value to get \( K \times K \) matrix \( S_K \), \( M \times K \) matrix \( U_K \) and \( N \times K \) matrix \( V_K \).

**Step 3** Get compressed matrix \( A_K \) (\( M \times N \)) by the formula \( A_K = U_K S_K V_K^T \).

**Step 4** Restore image by using the transformed matrix data, then update and display the image.

#### 4.2. Tensor approximation algorithm based on HOSVD

In previous part, we have proved that HOSVD is natural extension of matrix SVD and we can obtain n-mode singular value and singular vector according to HOSVD. It is prefer that singular vector corresponding to larger singular value conserve more energy in SVD. The fact is workable in HOSVD. Consequently, we can get tensor approximation algorithm based on HOSVD according to the following theorem in a similar way as the idea of truncated SVD [10, 18].

**Theorem 2:** In HOSVD, n-mode singular vectors corresponding to the first \( K_n \) largest n-mode singular values preserve the most energy of tensor \( A \).

Truncation version of mode-k unfolding matrix of tensor \( A \) results in an approximation \( \tilde{A} \) by discarding the smallest n-mode singular values \( \alpha_{k_1,1}^{(1)}, \alpha_{k_2,2}^{(2)}, \ldots, \alpha_{k_N, N}^{(N)} \) for given values of \( K_n \):

\[
\tilde{A} \approx \sum_{k_1}^{K_1} \sum_{k_2}^{K_2} \sum_{k_N}^{K_N} B(k_1, k_2, \ldots, k_N) U^{(1)}_{k_1} \circ U^{(2)}_{k_2} \circ \ldots \circ U^{(N)}_{k_N}
\]  

The error of the approximation can be obtain from equation (4-2).
$$\| A - \tilde{A} \| = \sum_{i=1}^{R_1} \sum_{j=1}^{R_2} \sum_{k=1}^{R_3} \alpha_{ij} \leq \sum_{i=1}^{R_1} \alpha_i^2 + \sum_{j=1}^{R_2} \alpha_j^2 + \ldots + \sum_{k=1}^{R_3} \alpha_k^2 \quad (4-3)$$

In which $R_n$ is the rank of unfolding matrix $A_{(n)}$. It is clear to see that the error is bounded by the sum of squared singular values associated with the discarded singular vectors. An obvious way to calculate minimal decompositions is to minimize the Frobenius norm of the difference between the right and left hand sides of (4-1) for increasing values of $r$, until that point in which the minimum vanishes. Truncation of the n-mode matrices resulting from the n-mode SVD algorithm may yield a good approximation, but it is generally not optimal. Instead of the direct computation of HOSVD, an alternating least squares (ALS) approach is most often used to obtain an optimal approximation of tensor.

Algorithm: tensor approximation based on high order singular value decomposition (THOSVD)

Input: given tensor data $X \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$

Output: Compressed tensor data $Y \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$

Step 1: Initialize project matrices $U_{(n)} \in \mathbb{R}^{I_n \times P_n}$, $n = 1, 2, \ldots, N$.

Step 2: for $t=1, 2, \ldots, \text{Tmax}$ do

for $n=1, 2, \ldots, N$ do

$X = X \times_1 U_{(1)}^{(t)} \times_2 U_{(2)}^{(t)} \times_3 \ldots \times_{n-1} U_{(n-1)}^{(t)} \times_n U_{(n)}^{(t)} \times_{n+1} U_{(n+1)}^{(t)} \times_{n+2} \ldots \times_N U_{(N)}^{(t)}$

$U_{(n)}^{(t)} = \text{svd}(X_{(n)});$

end

$Y = X \times_1 U_{(1)}^{(t)} \times_2 U_{(2)}^{(t)} \times_3 \ldots \times_{N-1} U_{(N-1)}^{(t)} \times_N U_{(N)}^{(t)}$;

if $||Y|| - ||Y_{(t)}|| < \varepsilon$ break;

end

5. Experiments

In order to demonstrate the efficiency of the algorithm, comparison experimentation is processed with JPEG compression. Three different kinds of images with the size of $512 \times 512$ shown in Figure 1 are chosen.

**Figure 1.** Sample images for compression

From the perspective of informatics, the information of Baboon image mainly lie on high frequency, Lena image contains more middle frequency information and Pepper image lies on low frequency. We use

To estimate the quality of reconstruction image under variable compression radio as shown in Figure 2.
The following conclusions can be drawn from Figure 2 that PSNR decrease with the increase of compression ratio for both JPEG and THOSVD compression. The fact originates that when compression ratio becomes larger, loss of information become more which results in worse reconstruction quality.

From Figure 3 to Figure 5, some reconstruction images are indicated by using JPEG and HOSVD with high compression ratio.

It is apparent that the quality of three different kinds of reconstruction image is consistent with
PSNR. We can draw the following conclusions from Figure 3 to Figure 5:

1. For middle frequency image, when compression radio approach 1, PSNR of reconstruction image based on JPEG outperform those based on THOSVD.

2. For Baboon and Peppers, the compression algorithm always has better reconstruction quality than JPEG compression.

3. When compression radio is high, reconstruction images based on JPEG have obvious block effect, but those images based on THOSVD have not the phenomenon.

6. Conclusions

Inspired by High Order Singular Value Decomposition, a novel image compression algorithm has been proposed in the paper. Firstly, we have processed theoretical analysis and shown that HOSVD is naturally extension of matrix SVD. Then, considering n-mode singular vector corresponding to the larger n-mode singular value in tensor decomposition capture more energy, we introduce tensor approximation algorithm based on high order singular value decomposition and apply it to realize image compression. The comparison experiment shows that tensor approximation algorithm based on HOSVD has better performance than JPEG in image processing.

7. References