The Design of High Attack-angle Flying Missile Control System Based on Feedback Linearization

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Abstract

During intercepting the fast targets or the maneuvering targets, the air-to-air missile often need to do large angle maneuver. But in this time, the characteristics of the missile control system become strong-nonlinearity and strong-coupling. And also, using traditional Euler Angles description method, the motion equation of a missile will be singular when the attack angle is big. In this paper the unit quaternion is used to describe the missile attitude and on the basis, the missile attitude control model is linearized by using feedback linearization method. Then sliding mode variable structure controller is designed for the linearization system. Finally, the correctness and the robustness of the control strategy are validated by the simulation results.

Keywords: Quaternion, Feedback Linearization, Missile Attitude Control, Variable-Structure-Control, Simulation

1. Introduction

In recent years, with the development of the technology of attack-defense confrontation, the mobility of the air targets (e.g. aircraft and missile) have been greatly improved. In order to intercept the air targets effectively, the air-to-air missile must have a better mobility. From the existing references [1-2] we can see: if we can increase the attack angle of the missile, the overload of the missile will be greatly improved, so as to achieve the tactics and technology index of the high-mobility missile.

However in the high-attack-angle flying conditions, the aerodynamic characteristics of the missile will become very complicated, even a very small change in missile attitude may also produce large aerodynamic and large aerodynamic moment, the characteristics of the missile control system become strong-nonlinearity and strong-coupling. Therefore using nonlinear control strategy is the inevitable way to control the high attack-angle flying missile. Feedback linearization control method is a kind of nonlinear control design method which has attracted a great deal of research interest in recent years[3-5], it can shield the nonlinear characteristics of the system so that the closed-loop dynamics are in a linear form.

In references [3-5], the traditional Euler Angle method was used to describe the attitude of the missile. When the pitch angle of the missile is 90°, the motion equation of the missile described by Euler Angle method will be singular, therefore, using the traditional Euler Angle method to describe the high-attack-angle missile is inappropriate that it can not satisfy the precision and stability requirement of the missile system [6]. In recent years, with the rapid development of computer technology, quaternion method is widely used in describing the missile attitude [7-9]. In this paper the unit quaternion is used to describe the missile attitude and on this basis, the motion equation of the missile described by quaternion method is established. Then the nonlinear missile attitude control system is linearized by using feedback linearization method, the original nonlinear system is divided into a six-order linear subsystem and a first-order internal dynamic nonlinear subsystem, then an analysis is made to find out the relationship between the first-order internal dynamic nonlinear subsystem and the stability of the whole system. After that, the sliding mode variable structure controller based on modified exponential approach law is designed for the six-order linear subsystem. Finally, the correctness and the robustness of the control strategy are validated by the simulation results.

2. Missile attitude motion equation

Spacecraft control system is a nonlinear multi-input multi-output system, its attitude dynamics equation can be expressed as:

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where $w_x$, $w_y$, $w_z$ are the rotational angular velocity of the missile respectively under the body coordinate system, $J_x, J_y, J_z$ are the moment of inertia of the missile respectively under the body coordinate system, $M_x, M_y, M_z$ are the moment produced by all the external force of the missile respectively under the body coordinate system.

If we use the traditional Euler Angle method to describe the kinematics equation of the missile, the equation can be expressed as:

$$
\begin{align*}
\dot{\gamma} &= w_x + (-w_y \cos \gamma + w_z \sin \gamma) \tan \theta \\
\dot{\theta} &= w_x \sin \gamma + w_y \cos \gamma \\
\dot{\varphi} &= (w_z \cos \gamma - w_x \sin \gamma) / \cos \theta
\end{align*}
$$

(2)

Where $\gamma, \theta, \varphi$ are the roll angle, pitch angle and the yaw angle of the missile respectively. From (2), we can see in some special conditions, equation (2) will become singular, and the attitude of the missile will be uncertain (e.g. when $\theta = 90^\circ$, the yaw angle $\varphi$ will be uncertain). Therefore this modeling method is not suitable for describing the large angle attitude motion of the missile. Meanwhile, when the missile is making large angle maneuvering, the nonlinear coupling of the three channels in the missile attitude control system will be enhanced, using the traditional small disturbance linearization method to design the missile attitude control system will lead to large errors inevitably, and even cause the missile out of control.

Using the unit quaternion method to describe the kinematics equation of the missile, the equation can be expressed as:

$$
\begin{bmatrix}
q_0 \\
q_1 \\
q_2 \\
q_3
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
q_0 - q_1 - q_2 - q_3 \\
q_1 - q_0 - q_3 + q_2 \\
q_2 - q_3 - q_0 + q_1 \\
q_3 - q_2 - q_1 + q_0
\end{bmatrix} \begin{bmatrix}
w_x \\
w_y \\
w_z
\end{bmatrix}
$$

(3)

where $Q = (q_0, q_1, q_2, q_3)^T$ is the missile attitude described by the unit quaternion and $q_0, q_1, q_2, q_3$ satisfy:

$$
q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1
$$

(4)

If we suppose that $q_0 = \cos \eta$, then $\sqrt{q_1^2 + q_2^2 + q_3^2} = \sin \eta$.

The relationships between the quaternion and the Euler angle are as follows:

$$
\sin \theta = 2(q_1 q_2 + q_0 q_3)
$$

(5)

$$
\tan \varphi = \frac{-2(q_1 q_3 - q_0 q_2)}{q_0^2 + q_1^2 - q_2^2 - q_3^2}
$$

(6)

$$
\tan \gamma = \frac{-2(q_2 q_3 - q_0 q_1)}{q_0^2 - q_1^2 + q_2^2 - q_3^2}
$$

(7)
If we suppose $u, x, y$ are the input, state vector and the output of the missile attitude control system respectively, and

$$
\begin{align*}
    u &= (u_1, u_2, u_3)^T = \begin{bmatrix} M_x & M_y & M_z \end{bmatrix}^T \\
    x &= (x_1, x_2, x_3, x_4, x_5, x_6, x_7)^T \\
    y &= (w_1, w_2, w_3, q_0, q_1, q_2, q_3)^T \\
    y &= h(x) = (h_1(x), h_2(x), h_3(x))^T \\
    y &= (x_5, x_6, x_7)^T = (q_1, q_2, q_3)^T
\end{align*}
$$

Then the equation of missile attitude motion described by (1) and (3) can be expressed as:

$$
\begin{align*}
    \dot{x} &= f(x) + g(x)u \\
    y &= h(x)
\end{align*}
$$

(8)

Where

$$
\begin{align*}
    f(x) &= \begin{bmatrix}
        0 & -J_z & J_y & 0 & 0 & 0 & -x_2x_3 \\
        J_z & 0 & -J_x & 0 & 0 & 0 & -x_1x_2 \\
        -J_y & J_x & 0 & 0 & 0 & 0 & -x_3x_1 \\
        0 & 0 & 0 & -J_z & J_y & -x_1x_2 & 0 \\
        0 & 0 & 0 & J_z & -J_x & x_3x_1 & 0 \\
        0 & 0 & 0 & -J_y & J_x & x_2x_3 & 0 \\
        -0.5(x_4x_1 + x_6x_2 + x_7x_3) & 0.5(x_4x_1 - x_7x_2 + x_6x_3) & 0.5(x_4x_1 + x_6x_2 - x_7x_3) & 0.5(-x_6x_1 + x_5x_2 + x_4x_3)
    \end{bmatrix},
    g(x) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
        0 & 1 & 0 & 0 & 0 & 0 & 0 \\
        0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{align*}
$$

3. Feedback linearization of the missile attitude control system

Feedback linearization is an approach to nonlinear control design. The central idea of the approach is to algebraically transform a nonlinear system dynamics into a (fully or partly) linear one, so that linear control techniques can be applied [10]. In its simplest form, feedback linearization amounts to canceling the nonlinearities in a nonlinear system so that the closed-loop dynamics are in a linear form.

Feedback linearization can be used to make a kind of nonlinear system to be linear, this kind of system is to be said linear in control or affine, it can be expressed as:

$$
\begin{align*}
    \dot{x} &= f(x) + g(x)u \\
    y &= h(x)
\end{align*}
$$

(9)

Obviously, the missile control system described by (8) is an affine nonlinear system, so we can use feedback linearization method to canceling the nonlinearities in the system.

Suppose the equilibrium point of the system is: $x_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T$, after calculation we can get that in the equilibrium point, $r_1 = r_2 = r_3 = 2$, where $r_1, r_2, r_3$ are the relative degree of the subsystems respectively, that means in the equilibrium point, the relative degree of the system is $r = 6$. From the state equation of the system, we can see the dimension of the system is $n = 7$, obviously, $r < n$. Therefore the system equation described by the unit quaternion can not be linearized accurately. After linearization, the state vector of the system will be divided to be a six-dimension linear state and a one-dimension internal dynamic state. The internal dynamic state is nonlinear and unobservable, it is affected by all the input and all the state of the system, but it does not affect the output of the system. Therefore if the internal dynamic is stable, it will not affect the stability of the whole system [11], then we can only need to design the linear subsystem.
3.1. Input transformation

Because \( r_1 = r_2 = r_3 = 2 \), according to the definition of relative degree, we can differentiate the three components of the output \((r_1, r_2, r_3)\) times respectively, then the relationship between input \(u\) and output \(y\) will be appears:

\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\dot{y}_3
\end{bmatrix} = A(x) + B(x)u
\]  

(10)

where \( A(x) = \begin{bmatrix}
L^2_1 h_1(x) & L^2_2 h_2(x) & L^2_3 h_3(x)
\end{bmatrix}^T \), \( B(x) = \begin{bmatrix}
L^3_1 L_f h_1(x) & L^3_2 L_f h_2(x) & L^3_3 L_f h_3(x)
\end{bmatrix} \).

\( L_f h_i(x) \) is called Lie derivative, it is simply the directional derivative of \( h_i(x) \) along the direction of the vector \( f(x) \), \( L_f h_i(x) = \frac{\partial h_i(x)}{\partial x} \).

\( L^2_f h_i(x) \) is to repeat Lie derivative twice. It can be expressed as \( L^2_f h_i(x) = L_f(L_f h_i(x)) \).

\( L^3_f h_i(x) \) is the scalar function which can be expressed as: \( L^3_g L_f h_i(x) = L^3_g(L_f h_i(x)) \).

If we suppose \( v = A(x) + B(x)u \), we can obtain that:

\[
\dot{\gamma} = \begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\dot{y}_3
\end{bmatrix} = v
\]  

(11)

Suppose that \( \gamma \) is the new input of the system, then we get the linear relationship between input and output. According to the generalized inverse technology, we can obtain:

\[
u = -B(x)\dot{\gamma} - B(x)B(x)^T \gamma^{-1}(v - A(x))\]  

(12)

3.2. State transformation

Because \( \mu_i = (\mu_1, \mu_2)^T = (y_i, \dot{y}_i)^T = (h_i(x), L_f h_i(x))^T (i = 1, 2, 3) \), according to the definition of the relative degree of the system, we can easily get the state transformation of the system. The state vector of the six-order linear subsystem is:

\[
\mu = \begin{bmatrix}
y_1 \\
\dot{y}_1 \\
y_2 \\
\dot{y}_2 \\
y_3 \\
\dot{y}_3
\end{bmatrix}^T
\]

This six-order linear subsystem is observable and controllable. We only need to discuss the stability of the one-order internal dynamic subsystem.

In order to make

\[
z = \phi(x) = \begin{bmatrix}
\mu^T \\
\psi^T
\end{bmatrix} = \begin{bmatrix}
y_1 \\
\dot{y}_1 \\
y_2 \\
\dot{y}_2 \\
y_3 \\
\dot{y}_3
\end{bmatrix}^T
\]

to be a state transformation, the internal dynamic \( \psi \) need to satisfy:

\[
L_g \psi = \frac{\partial \psi}{\partial x} \mu = 0
\]

(13)

If we suppose \( \psi = q_0 \), we can see, \( L_g x_2 = L_g q_0 = 0 \), that means \( \psi = q_0 \) can satisfy (13). Furthermore, we can prove that the vector field \( \Lambda = span\{g_1, g_2, g_3\} \) is involutive, so we can choose \( \psi = q_0 \) to be the internal dynamic of the system.

Suppose

\[
z = \phi(x) = \begin{bmatrix}
\mu^T \\
\psi^T
\end{bmatrix} = \begin{bmatrix}
y_1 \\
\dot{y}_1 \\
y_2 \\
\dot{y}_2 \\
y_3 \\
\dot{y}_3
\end{bmatrix}^T
\]
while its Jacobian is:

\[
\frac{\partial z}{\partial x} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0.5x_4 & -0.5x_7 & 0.5x_6 & 0.5x_1 & 0 & 0.5x_3 & -0.5x_2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0.5x_5 & 0.5x_4 & -0.5x_3 & 0.5x_2 & -0.5x_1 & 0 & 0.5x_0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-0.5x_6 & 0.5x_5 & 0.5x_4 & 0.5x_3 & 0.5x_2 & -0.5x_1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

It is nonsingular at every point \(x\), so \(z = \phi(x) = \begin{bmatrix} \mu \ y \end{bmatrix}^T\) is a local coordinate transformation, and its inverse transformation exists. Let \(\phi^{-1}(z)\) denote the inverse transformation, we can get:

\[
x = \phi^{-1}(z) = \phi^{-1}(\mu, y)
\]  

(14)

From (8), we can obtain:

\[
\dot{q}_0 = -0.5(x_5x_1 + x_6x_2 + x_7x_3)
\]  

(15)

Using the new state vector, we can get:

\[
\dot{y} = \dot{q}_0 = -0.5(x_5x_1 + x_6x_2 + x_7x_3)\big|_{\phi^{-1}(\mu, y)}
\]  

(16)

One easily finds that:

\[
\dot{y} = -\frac{1}{y'}(\mu_1 \mu_2 + \mu_3 \mu_4 + \mu_5 \mu_6)
\]  

(17)

Let us think about the zero dynamic of the system:

\[
\dot{y}(0, y') = -\frac{1}{y'} \times 0
\]  

(18)

As long as \(y'(0) \neq 0\), \(\dot{y}(0, y') = 0\) can be always satisfied. That means the zero dynamic of the system is stable, then this nonlinear system is a minimum phase system.

According to equation (4), under the condition of \(y' = q_0 \neq 0\), we can get the solution of \(y'\):

\[
y' = q_0 = \cos \eta
\]  

(19)

where \(\eta\) is an angle. That means, as long as \(y'(0) \neq 0\) and the control curve does not reach to the points \(y' = 0\), we can ensure that no matter how state vector \(\mu\) changes, the internal dynamic will always be stable. After feedback linearization, the state vector of the system is:

\[
z = (q_1 \quad \dot{q}_1 \quad q_2 \quad \dot{q}_2 \quad q_3 \quad \dot{q}_3 \quad \psi)^T
\]  

(20)

4. Design of the controller

According to (4), \(q_0, q_1, q_2, q_3\) are not independent of each other, if we know any three of them, the other one can be obtained. So we only need to consider three of them when designing the controller. In this paper, we consider \(q_1, q_2, q_3\).
After feedback linearization, the original nonlinear system is divided into a six-order linear subsystem and a first-order internal dynamic nonlinear subsystem, from the previous section, we have proved that the internal dynamic is stable, so we only need to design the six-order linear subsystem.

The state equation of this linear subsystem can be expressed as:

$$\dot{\mu} = A\mu + B\nu \quad y = C\mu$$  \hspace{1cm} (21)$$

where

$$A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (i = 1,2,3),$$

From above we can see the output $y_i$ is only affected by the input $\mu_i$, there is no relationship between $y_i$ and other inputs $\mu_j (j \neq i)$. That means the six-order linear system is decoupled into three independent subsystems. We can devise them respectively. In this paper, the sliding mode variable structure control technology is used to design these three linear subsystems.

The biggest advantage of sliding mode variable structure controller is that according to the uncertainties of the system, the sliding movement has strong robustness. When the uncertainties can satisfy the matching conditions, the sliding movement will be completely free from these uncertainties, even if the matching conditions can not be satisfied, the sliding movement can still has strong robustness and good dynamic performance by devise the sliding surface properly [11]. For the variable structure controller design, there mainly includes two steps: (1) design of switching function $s$, (2) design of variable structure control law.

4.1. Design of switching functions

Suppose the command output of the missile is $y_c = [q_{1c} \quad q_{2c} \quad q_{3c}]^T$ (in the inertial coordinate system), let

$$e_i = y_i - y_{ic}$$
$$\dot{e}_i = \dot{y}_i - \dot{y}_{ic} \quad (i = 1,2,3)$$

be the error between the command output and the actual output. Let $s_i(e_i)$ to be the switching surface, where

$$s_i(e_i) = \left( \frac{d}{dt} + c \right)^{n_i-1} e_i$$  \hspace{1cm} (23)$$

According to the previous section, the three linear subsystems are 2-order ($n_i = 2$), so $s_i(e_i)$ can be expressed as:

$$s_i(e_i) = \dot{e}_i + c_i e_i$$  \hspace{1cm} (24)$$
Equation (24) is the expression of switching function, let \( s_i(e_i) \) to be zero, we can get the equation of switching surface:

\[
s_i(e_i) = \dot{e}_i + c_i e_i = 0 \tag{25}
\]

Let the lyapunov function to be:

\[
V_i = \frac{1}{2} s_i^2(e_i) \tag{26}
\]

and its derivation is

\[
\dot{V}_i = s_i \dot{s}_i = s_i \times (\ddot{e}_i + c_i \dot{e}_i) \tag{27}
\]

As long as \( c_i \) is appropriate, the derivation of \( V_i \) can satisfy \( \dot{V}_i < 0 \), then \( s_i \) will tend to be zero.

### 4.2 Design of variable structure control law

The key technology of making the switching surface to be the sliding surface is the design of variable structure control law. The law can make the system reach the switching surface under any initial state in a finite time. Since the implementation of the associated control switching is necessarily imperfect (for instance, in practice switching is not instantaneous, and the value of s is not known with infinite precision), this leads to chattering. There were many methods to solve this problem, such as continuous sliding mode control and reaching law sliding mode control. In this paper, we choose the reaching law sliding mode control technology to design the controller and choose the reaching law to be exponential approaching rule:

\[
\dot{s}_i = -c_i \text{sgn } s_i - k_i s_i \tag{28}
\]

where \( c_i, k_i \) are both constants greater than zero. \text{sgn} expresses the sign function.

Then we can get the variable structure control law which is expressed as:

\[
v_i = \frac{1}{c_i + k_i} \{ \ddot{y}_w + (c_i + k_i) \dot{y}_w + k_i y_i - \dot{y}_i - k_i c_i y_i + e_i \text{sgn } s_j \} \tag{29}
\]

After that, the initial input \( u \) can be obtained from equation (12).

### 5. Simulation

In order to verify the effectiveness of the method in this paper, we will make a simulation which uses this method to control a large angle flying air-to-air missile. Suppose the nominal values of the moment of inertias are \( J_{x,0} = 0.4 \text{kg} \cdot \text{m}^2 \), \( J_{z,0} = J_{y,0} = 3 \text{kg} \cdot \text{m}^2 \). The initial values and the instruction values of the system are expressed as:

\[
y_0 = (0 \ 0 \ 0)^T ; \ w_0 = (0 \ 0 \ 0)^T ; \ y_c = (0 \ 0 \ \sqrt{2}/2)^T ; \ w_c = (0 \ 0 \ 0)^T ; \ k_i = 5 ; \ c_i = 1 ; \ e_i = 10^{-5} \ (i = 1, 2, 3) .
\]

From these initial values and the instruction values, we can obtain the initial and the instruction attitudes of the missile described by unit quaternion are respectively expressed as:

\[
Q_0 = (1 \ 0 \ 0) ; \ \ Q_c = (\sqrt{2}/2 \ 0 \ 0 \ \sqrt{2}/2)^T .
\]

According to equation (5), (6), (7), we can obtain that the initial and the instruction attitudes described by Euler angle are \( \theta_0 = 0^\circ \), \( \theta_c = 90^\circ \), that means the missile need to adjust its attitude for 90°.
Figure 1 shows the simulation result. From the figure we can see that the attitude of the missile can be controlled very well.

![Figure 1. The tracking curve of the missile attitude](image)

If we add attitude disturbance at time points $t = 3s$ and $t = 7s$, figure 2 shows the error curve of the attitude tracking system, from this figure, we can see the control system in this paper has a good inhibitory effect on the disturbance, and the accuracy of the system is very good.

![Figure 2. The tracking error curve of the missile attitude](image)

6. Conclusions

In order to solve the control problem of the strong-nonlinearity and strong-coupling missile control system, the unit quaternion was used to describe the missile attitude and on this basis, the motion equation of the missile described by quaternion method was established. Then the nonlinear missile attitude control system was linearized by using feedback linearization method, the original nonlinear system was divided into a six-order linear subsystem and a first-order internal dynamic nonlinear subsystem, then an analysis was made to find out the relationship between the first-order internal dynamic nonlinear subsystem and the stability of the whole system. After that, the sliding mode variable structure controller based on modified exponential approach law was designed for the six-order linear subsystem. The simulation results show that under this control method, the missile can be controlled even if the pitch angle is $90^\circ$ successfully, and the correctness and the robustness of the control strategy are validated by the simulation results.

7. Reference


