Analytical Approximate Solutions of the Benney-Lin Equation with Symbolic Computation

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Abstract

With the aid of the symbolic computation software Maple, the paper deals analytically with approximate solutions of the Benney-Lin equation. A direct technique, homotopy analysis method, is employed to solve the Benney-Lin equation with the initial-value condition. A comparison is made between the approximate solutions and exact solutions, which reveals that the approximate solutions are accurate and effective. The results prove that the employed method is a reliable and powerful mathematical tool to solve nonlinear differential equation.

Keywords: Benney-Lin Equation, Approximate Solution, Symbolic Computation.

1. Introduction

Nonlinear differential equations attract a huge size of research work to establish their exact and approximate solutions. With the availability of computer systems like Maple and Mathematica which allow us to perform some complicated algebraic and differential calculation on a computer, many powerful methods have been presented to generate solutions of nonlinear differential equations[1-36]. Among them, the homotopy analysis method is one of the strong tools and is proposed by Liao [1-3]. This technique has been applied successfully to solve many types of nonlinear problems [4-8].

The present paper concerns the Benney-Lin equation [9,10] with the form

\[ u_t + uu_x + u_{xxx} + \beta (u_{xx} + u_{xxxx}) + \eta u_{xxxx} = 0, \]  

where \( \beta, \eta \) are constants. When \( \beta = 0 \), Eq. (1) is a Kawahara equation (or a fifth-order KdV equation) which can describe the plasma waves and the capillary-gravity water waves [11]. Some approaches have been used to obtain exact solutions of the Kawahara equation, such as the G'/G - expansion method [12], the Exp-function method [13], the sine-cosine method [14], and so on. The approximate solutions of the Kawahara equation are given using the the homotopy analysis method [15] and the optimal homotopy-analysis method [16]. If let \( \eta = 0 \), Eq. (1) is a generalized Kuramoto-Sivashinsky equation (or a KdV-Burgers-Kuramoto equation) which models long waves on a viscous fluid flowing down along an inclined plane [17] and unstable drift waves in plasma [18]. The generalized Kuramoto-Sivashinsky equation has been studied using many methods including tanh function method [19], local discontinuous Galerkin methods [20], Chebyshev spectral collocation methods [21] and Lattice Boltzmann method [22]. Berloff [23] employed the singular manifold method and partial fraction decomposition to find solitary and periodic solutions of the Kawahara equation and the generalized Kuramoto-Sivashinsky equation. The Benney-Lin equation has a prominent position in describing the evolution of small but finite amplitude long waves in fluid dynamics. Numerical solution of the Benney-Lin equation with the initial-boundary-value condition was investigated by Sepúlveda and Vera [24]. Xie [25] presented a combination method to construct the explicit and exact solutions of the Benney–Kawahara-Lin equation. Safari et al. [26] applied the homotopy perturbation method and the variational iteration method to the Benney-Lin equation. Gupta [27] obtained approximate solutions of fractional Benney–Lin equation using the reduced differential transform method and the homotopy perturbation method. The aim of this work is to employ the homotopy analysis method to solve the analytical approximate solutions of Eq. (1) and give the exact solutions by the Ricatti equation expansion method.
2. Description of the method

Consider the general differential equation with a physical field \( u(x,t) \),
\[
DE[u(x,t)] = 0. \tag{2}
\]

The first step of the homotopy analysis method [1-3] is to choose a proper linear operator and then construct the so-called zero-order deformation equation
\[
L[U(x,t;q) - u_0(x,t)] = q h H(x,t) DE[U(x,t;q)] \tag{3}
\]
where \( L \) is an auxiliary linear operator, \( q \in [0,1] \) is an embedding parameter, \( h \) is a nonzero auxiliary parameter, \( H(x,t) \) is an auxiliary function, \( u_0(x,t) \) is an initial guess of \( u(x,t) \), \( U(x,t;q) \) is a unknown function of independent variables \( x,t,q \).

Liao expanded \( U(x,t;q) \) in Taylor series with respect to \( q \) and gave
\[
U(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \tag{4}
\]
where
\[
u_{m}(x,t) = \frac{\partial^m U(x,t;q)}{\partial q^m} \bigg|_{q=0}. \tag{5}
\]

If the auxiliary operator, the initial guess, the auxiliary parameter and and the auxiliary function are properly chosen, the series (4) converges at \( q = 1 \), one has
\[
u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{6}
\]

Differentiating Eq. (3) \( m \) times with respect to the embedding parameter \( q \) and then dividing them by \( m! \) and finally setting \( q = 0 \), one has the so-called \( m \) th-order deformation equation
\[
L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = h H(x,t) RDE[u_{m-1}(x,t)], \tag{7}
\]
Where
\[
RDE[u_{m-1}(x,t)] = \frac{1}{(m-1)!} \frac{\partial^{m-1} DE[U(x,t;q)]}{\partial q^{m-1}} \bigg|_{q=0}, \tag{8}
\]
and
\[
\chi_m = \begin{cases} 
0 & m \leq 1, \\
1 & m > 1. 
\end{cases} \tag{9}
\]

The \( m \) th-order deformation equation (7) is linear and thus can be easily solved using the initial conditions, all components of series \( u_m \) are determinable by using Eq. (7). Substituting \( u_0, u_1, \ldots, u_m, \ldots \), in Eq. (6), \( u \) is obtained finally.

3. Approximate solutions

Consider Eq. (1) subject to the following initial condition
\[
u(x,0) = \varphi(x). \tag{10}
\]

Eq. (1) suggests
\[
DE[U(x,t;q)] = U_1(x,t,q) + U(x,t;q)U_1(x,t;q) + U_{xxx}(x,t;q)
+ \beta(U_{xx}(x,t;q) + U_{xxx}(x,t;q)) + \eta U_{xxxx}(x,t;q), \tag{11}
\]
and the linear operator is written as
\[
L[U(x,t;q)] = U_1(x,t;q), \tag{12}
\]
with the property

\[ L(C) = 0. \tag{13} \]

Based on the homotopy analysis method, the zeroth-order deformation equation is constructed with the assumption \( H(x,t) = 1 \),

\[ (1-q)L[U(x,t;q) - u_0(x,t)] = qhDE[U(x,t;q)]. \tag{14} \]

Then, we obtain the \( m \) th-order deformation equation

\[
L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hRDE[u_{m-1}(x,t)],
\]

\[
= h[u_{m-1,x} + \sum_{k=0}^{m-1} u_k u_{m-1-k,x} + u_{m-xxx} + \beta(u_{m-1xx} + u_{m-1xxx}) + \eta u_{m-1xxxxxxx}]. \tag{15}
\]

Using the above recursive relationship and the symbolic computation system Maple, the first few terms of the new decomposition series are given, namely

\[
u_0 = \phi(x),
\]

\[
u_1 = hL^{-1}[u_{1t} + u_0 u_{1x} + u_0_{xxx} + \beta(u_{1xx} + u_{0xxx}) + \eta u_{0xxxxx}]
= h[\phi' + \phi'' + \beta(\phi' + \phi^{(4)}) + \eta\phi^{(5)}],
\]

\[
u_2 = u_1 + hL^{-1}[u_{1t} + u_0 u_{1x} + u_1 u_{0x} + u_{1xxx} + \beta(u_{1xx} + u_{1xxx}) + \eta u_{1xxxx}]
= h(1 + h)[\phi' + \phi'' + \beta(\phi' + \phi^{(4)}) + \eta\phi^{(5)}] + \frac{h^2 t^2}{2} [(2\phi'' + \phi' + \eta\phi^{(5)}]
+ 2\eta\phi^{(6)} + \phi'' + (\phi'' + 4\phi'\beta)\phi' + 3\phi'^2 + (\beta + 5\phi' + 2\phi\beta + 10\beta\phi'')\phi''
+ 10\eta\phi''^2 + (1 + 6\phi'\beta + \beta^2 + 15\eta\phi''\phi^{(4)} + (7\eta\phi' + 2\beta\phi + 3\beta)\phi^{(5)} + (1 + 3\eta + 2\beta^2)\phi^{(6)} + 2\beta(1 + \eta)\phi^{(7)} + (2\eta + \beta^2)\phi^{(8)} + 2\beta\eta\phi^{(9)} + \eta^2\phi^{(10)}],
\]

\[
u_3 = u_2 + hL^{-1}[u_{2t} + u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x} + u_{2xxx} + \beta(u_{2xx} + u_{2xxx}) + \eta u_{2xxxx}]
= \ldots . \tag{16}
\]

The approximate solutions of Eq. (1) obtained by the homotopy analysis method are

\[ u = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \ldots . \tag{17} \]

4. Comparison and discussion

With the homotopy analysis technique, the results (17) contain an auxiliary parameter \( h \) which provides us with a simple way to adjust and control the convergence region of the series solutions. To establish the effective parameter \( h \), one can investigate the influence of \( h \) on the solutions by plotting the so-called \( h \)-curve. According to Liao’s theory, the appropriate region for the parameter \( h \) is an interval, where the \( h \)-curve corresponds to the line segment nearly parallel to the horizontal axis. In this section, the comparison is made between the approximates solutions with the valid values of the parameter \( h \) and exact solutions involved in Appendix.

Case \( \beta = \frac{8\sqrt{33}}{165} \) and \( \eta = \frac{12}{605} \)

Let \( \phi(x) = \frac{53}{36} + \lambda - \frac{14}{3} \tanh\left(\frac{\sqrt{33}x}{12}\right) - \frac{7}{6} \tanh^2\left(\frac{\sqrt{33}x}{12}\right) \n
Analytical Approximate Solutions of the Benney-Lin Equation with Symbolic Computation
Lina Song

805
The exact solution of this system (1) are
\[ \phi(x) = \frac{53}{36} + \frac{14}{3} \tanh\left(\frac{\sqrt{33} x}{12} - \lambda \right) - \frac{7}{6} \tanh^2\left(\frac{\sqrt{33} x}{12} - \lambda \right) + \frac{14}{3} \tanh^3\left(\frac{\sqrt{33} x}{12} - \lambda \right) - \frac{7}{4} \tanh^4\left(\frac{\sqrt{33} x}{12} - \lambda \right). \] (19)

The \( h \)-curves of \( u_x(3,0), u_y(3,0) \) given by the fifth-order approximate solutions of Eq. (1) with the condition (18) are drawn in Fig. 1, when \( \lambda = 1 \). It is clear that the series solution is convergent when \(-1.3 < h < -0.6\). Under the same conditions, Fig. 2 gives the comparison between the fifth-order approximations and exact solutions (19), which shows the approximation with \( h = -1.04 \) is more close to the exact value than the one with \( h = -0.96 \).

**Fig. 1.** The \( h \)-curves of \( u_x(3,0), u_y(3,0) \) for the fifth-order approximations with \( \lambda = 1 \). Solid curve: \( u_x(3,0) \); Dotted curve: \( u_y(3,0) \).

**Fig. 2.** The solution curves of Eq. (1) with \( x = 3 \) and \( \lambda = 1 \). Solid line: exact solution; Dotted line: approximate solution with \( h = -0.96 \); Dashed line: approximate solution with \( h = -1.04 \).
Case $\beta = \frac{22\sqrt{29}}{179}$ and $\eta = \frac{29}{179}$

Let

$$
\phi(x) = \frac{735}{5191} + \lambda - \frac{105}{5191(\cosh(\frac{x}{\sqrt{29}}) - 1)^2} [7 \cosh^2(\frac{x}{\sqrt{29}}) - 8 \cosh(\frac{x}{\sqrt{29}}) \sinh(\frac{x}{\sqrt{29}}) \\
+ 2 \cosh(\frac{x}{\sqrt{29}}) - 5].
$$

(20)

The exact solution of this system (1) are

$$
\phi(x) = \frac{735}{5191} + \lambda - \frac{105}{5191(\cosh(\frac{\lambda t}{\sqrt{29}}) - 1)^2} [7 \cosh^2(\frac{x-\lambda t}{\sqrt{29}}) \\
- 8 \cosh(\frac{x-\lambda t}{\sqrt{29}}) \sinh(\frac{x-\lambda t}{\sqrt{29}}) + 2 \cosh(\frac{x-\lambda t}{\sqrt{29}}) - 5].
$$

(21)

Fig. 3. The $h$ -curves of $u(2,0.01), u(2.5,0.1), u(3,1)$ from the fifth-order approximations with $\lambda = \frac{1}{2}$. Solid curve: $u(2,0.01)$; Dotted curve: $u(2.5,0.1)$ Dashed line: $u(3,1)$. 
Case \( \beta = -\frac{i\sqrt{2794}}{4} \) and \( \eta = \frac{127}{22} \)

Let \( \phi(x) = \frac{1}{2032 \cos^4\left(\frac{\sqrt{11} x}{\sqrt{254}}\right)} \cdot \left[ 762 \lambda - 13860 - 18480 \left(\sin\left(\frac{\sqrt{11} x}{\sqrt{254}}\right) - i \cos\left(\frac{\sqrt{11} x}{\sqrt{254}}\right)\right) + 1155i \left(10 \sin\left(\frac{2 \sqrt{11} x}{\sqrt{254}}\right) + \sin\left(\frac{\sqrt{11} x}{\sqrt{254}}\right)\right) + (4620 + 1016 \lambda) \cos\left(\frac{2 \sqrt{11} x}{\sqrt{254}}\right) + 254 \lambda \cos\left(\frac{4 \sqrt{11} x}{\sqrt{254}}\right) \right] \). \( \text{(22)} \)

The exact solution of this system (1) are given by

\[
\begin{align*}
  u(x,t) &= \frac{1}{2032 \cos^4\left(\frac{\sqrt{11}(x - \lambda t)}{\sqrt{254}}\right)} \cdot \\
  &\left[ 762 \lambda - 13860 - 18480 \left(\sin\left(\frac{\sqrt{11}(x - \lambda t)}{\sqrt{254}}\right) - i \cos\left(\frac{\sqrt{11}(x - \lambda t)}{\sqrt{254}}\right)\right) + 1155i \left(10 \sin\left(\frac{2 \sqrt{11}(x - \lambda t)}{\sqrt{254}}\right) + \sin\left(\frac{\sqrt{11}(x - \lambda t)}{\sqrt{254}}\right)\right) \right] \
  &- i \cos\left(\frac{\sqrt{11}(x - \lambda t)}{\sqrt{254}}\right) + 1155i \left(10 \sin\left(\frac{2 \sqrt{11}(x - \lambda t)}{\sqrt{254}}\right) + \sin\left(\frac{\sqrt{11}(x - \lambda t)}{\sqrt{254}}\right)\right) 
\end{align*}
\]
\[ + (4620 + 1016 \lambda) \cos \left( \frac{2\sqrt{11}(x - \lambda t)}{\sqrt{254}} \right) + 254 \lambda \cos \left( \frac{4\sqrt{11}(x - \lambda t)}{\sqrt{254}} \right) \] 

(23)

Fig. 5. The surfaces of the real part of the solutions to Eq. (1) with \( \lambda = \frac{3}{2} \). (a) The approximations with \( h = -0.95 \); (b) Exact solution.

Fig. 5 gives the comparison of the real parts from the approximate solution and exact solution (23), when \( \lambda = \frac{3}{2} \). The comparison of the imaginary parts is shown in the Fig. 6. Figs. 5 and 6 illustrate the approximate solution with \( h = -0.95 \) is effective.

Fig. 6. The surfaces of the imaginary part of the solutions to Eq. (1) with \( \lambda = \frac{3}{2} \). (a) The approximations with \( h = -0.95 \); (b) Exact solution.
5. Conclusions

The basic goal of this work is to derive the analytical approximate solutions of the Benney-Lin equation using the homotopy analysis method with the aid of the symbolic computation. The homotopy analysis method provides us with a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference between the employed approach and other methods. The comparison of the solution curves between approximate solutions and exact solutions illustrate that the results are effective and the homotopy analysis method is a promising tool for solving more complicated nonlinear differential equation systems.

6. Appendix

The tanh-function method [28] is one of the most straightforward and effective algebraic algorithms to obtain exact solutions for lots of nonlinear problems, from which the extended function expansion methods [19,29-34] evolve gradually. In the paper, we use a simple ansatz [19] and the Riccati equation to give exact solutions of Eq. (1). For that, the exact solution of Eq. (1) is expressed by the following ansatz:

$$u = \sum_{i=0}^{n} a_i \phi^i(\xi), \quad \xi = k(x - \lambda t), \quad \phi(\xi)$$  \hspace{1cm} (A.1)

where \( \phi(\xi) \) satisfies

$$\phi'(\xi) = e_0 + e_1 \phi^2(\xi). \quad \hspace{1cm} (A.2)$$

Eq. (A.2) has the following solutions

If \( e_0 = \frac{1}{2} \) and \( e_1 = -\frac{1}{2} \), then \( \phi = \tanh(\xi) \pm i \sec h(\xi) \) or \( \phi = \coth(\xi) \pm \csc h(\xi) \).

If \( e_0 = \frac{1}{2} \) and \( e_1 = \frac{1}{2} \), then \( \phi = \tan(\xi) \pm \sec h(\xi) \) or \( \phi = \cot(\xi) \pm \csc(\xi) \).

If \( e_0 = 1 \) and \( e_1 = -1 \), then \( \phi = \tanh(\xi) \) or \( \phi = \coth(\xi) \).

If \( e_0 = 1 \) and \( e_1 = 1 \), then \( \phi = \tan(\xi) \).

If \( e_0 = 1 \) and \( e_1 = -1 \), then \( \phi = \cot(\xi) \).

If \( e_0 = 0 \) and \( e_1 \neq 0 \), then \( \phi = \frac{1}{r_0 + r_1 \xi} \).

Balancing the highest nonlinear term and the highest-order partial derivative term in Eq. (1) gives \( n = 4 \). Substituting (A.1) along with (A.2) into (1) and setting the coefficients of \( \phi(\xi) \) to be zero yield a set of over-determined algebraic equations with respect to \( a_0, a_1, a_2, a_3, a_4, k, \lambda, \beta, \eta \).

Solving the over-determined algebraic equations get the following results:

Case 1

$$a_0 = \frac{53}{36} + \lambda, a_1 = \epsilon \frac{14e_1}{3} \sqrt{\frac{-1}{e_0 e_1}}, a_2 = \frac{7e_1}{6e_0}, a_3 = \epsilon \frac{14e_1^2}{3e_0} \sqrt{\frac{-1}{e_0 e_1}}, a_4 = -\frac{7e_1^2}{4e_0},$$

$$k = \epsilon \frac{\sqrt{33}}{12} \sqrt{\frac{-1}{e_0 e_1}}, \beta = \epsilon \frac{\sqrt{33}}{12} \sqrt{\frac{-1}{e_0 e_1}}, \eta = \frac{12}{605}.$$  \hspace{1cm} (A.3)

Case 2

$$a_0 = \frac{735}{5191} + \lambda, a_1 = \epsilon \frac{420e_1}{5191} \sqrt{\frac{-1}{e_0 e_1}}, a_2 = \frac{630e_1}{5191 e_0}, a_3 = -\epsilon \frac{420e_1^2}{5191 e_0} \sqrt{\frac{-1}{e_0 e_1}}, a_4 = -\frac{105e_1^2}{5191 e_0}. $$  \hspace{1cm} (A.4)
$$k = e^2 \frac{\sqrt{2}}{12} \sqrt{-\frac{1}{e_0 e_1}} \left(-e^{\frac{\sqrt{33}}{12}} \sqrt{-\frac{1}{e_0 e_1}}\right), \beta = -\frac{22\sqrt{29}}{179} \left(-\frac{22\sqrt{29}}{179}\right), \eta = \frac{29}{179}.$$ 

Case 3

$$a_0 = -\frac{1155}{1061} + \lambda, a_1 = e^{\frac{3465}{508}} \sqrt{\frac{1}{e_0 e_1}}, a_2 = -\frac{1155 e_1}{508 e_0}, a_3 = e^{\frac{1155 e_1^2}{508 e_0}}, a_4 = -\frac{1155 e_1^3}{1061 e_0^2}, k = e^{\frac{\sqrt{2794}}{508}} \sqrt{-\frac{1}{e_0 e_1}} \left(-e^{\frac{\sqrt{2794}}{508}} \sqrt{-\frac{1}{e_0 e_1}}\right),$$

$$\beta = -\frac{i \sqrt{2794}}{4} \left(-\frac{i \sqrt{2794}}{4}\right), \eta = \frac{127}{22}.$$

Where $e = \pm 1$. Solutions (A.1) with the values of the parameters $a_0, a_1, a_2, a_3, a_4, k, \lambda, \beta, \eta$ are exact solutions of Eq. (1). We can only list four cases in which the expression of the parameters is simpler.

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8. References


