A Class of L-Cospectral Graphs are Isomorphic
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Abstract

The spectral graph theory is a theory in which graphs are studied by means of their adjacency eigenvalues, Laplacian eigenvalues and signless Laplacian eigenvalues of associated graph matrix. What kind of graphs are determined by their spectra is a difficult problem in graph theory. Let $H_q(C_n, T(n_1, n_2, n_3))$ be a graph of order $n$ obtained by identifying a vertex of cycle $C_n$ with the vertex of degree three of T-shape $T(n_1, n_2, n_3)$. In this paper, we will prove that for $q$ even, all the L-cospectral graphs $H_q(C_n, T(n_1, n_2, n_3))$ are isomorphic.

Keywords: Spectra Of Graphs, Cospectral Graphs, Laplacian Spectrum

1. Introduction

We are concerned only with undirected simple graphs (loops and multiple edges are not allowed). Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Let matrix $A(G)$ be the adjacency matrix of $G$ and $d_v$ the degree of the vertex $v$. The matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of $G$. Let $P_{L(G)}(\mu) = |\mu I - L(G)|$ be the Laplacian characteristic polynomial of $G$, where $I$ is the identity matrix, and let $P_{L(G)}(\mu) = \mu_1^{l_1} \mu_2^{l_2} + \cdots + \mu_r^{l_r}$. Assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r (= 0)$ are the adjacency and the Laplacian eigenvalues of $G$, respectively. They compose the adjacency (Laplacian) spectrum, denoted by A-spectrum and L-spectrum, respectively. Spectral graph theory can be used in many fields, such as face recognition [10-11]. Two non-isomorphic graphs are said to be cospectral if they have equal spectrum [1][5]. A graph is said to be determined by its spectrum if there is no other non-isomorphic graph with the same spectrum [6-9].

In this paper, we present a class of L-cospectral graphs which are isomorphic. Let $C_q$ be a cycle of order $q$, $T(n_1, n_2, n_3)$ be a T-shape tree of order $n_1 + n_2 + n_3 + 1$ such that $T(n_1, n_2, n_3) - v = P_{n_1} \cup P_{n_2} \cup P_{n_3}$, where $v$ is the vertex of degree three. It is well known that $C_q$ and $T(n_1, n_2, n_3)$ are determined by their Laplacian spectra, respectively. Denote by...
$H_0\{C_q, T(n, n_2, n_3)\}$ a graph of order $n$ obtained by identifying a vertex of cycle $C_q$ with the vertex $v$ of $T(n, n_2, n_3)$. We will prove that for $q$ even, all the $L$-cospectral graphs $H_0\{C_q, T(n, n_2, n_3)\}$ are isomorphic.

2. Preliminaries

**Lemma 2.1** [1][2] Let $G$ be a graph. For the adjacency matrix and the Laplacian matrix, the following can be deduced from the spectrum.

1. The number of vertices.
2. The number of edges.
3. Whether $G$ is regular.

For the adjacency matrix, the following follows from the spectrum.

4. The number of closed walks of any length.
5. Whether $G$ is bipartite.

For the Laplacian matrix, the following follows from the spectrum.

6. The number of components.
7. The number of spanning trees.
8. The sum of the squares of degrees of vertices.

**Lemma 2.2**[3] Let $u$ be a vertex of $G$, $N(u)$ be the set of all vertices adjacent to $u$ and $C(u)$ be the set of all cycles containing $u$. The characteristic polynomial of $G$ satisfies

$$P_{A(G)}(\lambda) = \lambda P_{A(G-u)}(\lambda) - \sum_{v \in N(u)} P_{A(G-u-v)}(\lambda) - 2 \sum_{z \in C(u)} P_{A(G \setminus \{v\})}(\lambda).$$

For the sake of simplicity, we denote $P_{A(P^n)}(\lambda)$ by $p_r(\lambda)$.

**Lemma 2.3** $p_r(2) = P_{A(P^n)}(2) = r + 1$ and $P_{A(C^r)}(2) = 0$, where $P_r$ denotes the path with $r$ vertices and $C^r$ the cycle with $r$ vertices.

**Lemma 2.4**[2] Let $G$ be a graph with $n$ vertices and $m$ edges and let $\text{deg}(G) = (d_1, d_2, \ldots, d_n)$ be its non-increasing degree sequence. Then the first four coefficients in $P_{L(G)}(\mu)$ are:

$$l_0 = 1, \quad l_1 = -2m, \quad l_2 = 2m^2 - m - \frac{1}{2} \sum_{i=1}^{n} d_i^2,$$

$$l_3 = \frac{1}{3} (-4m^3 + 6m^2 + 3m \sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i^3 - 3 \sum_{i=1}^{n} d_i^2 + 6n_G(C_3)).$$

**Lemma 2.5**[4] If two bipartite graph $G$ and $G'$ of order $n$ are Laplacian cospectral, then their line graph $\Gamma(G)$ and $\Gamma(G')$ are adjacency cospectral.

3. Main results

**Lemma 3.1** Let $G$ be a connected unicyclic graph of order $n$ with its cycle $C_q$. If $G'$ is $L$-cospectral to $G$, then $G'$ must be a connected unicyclic graph of order $n$ with its cycle $C_q$. 

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Moreover, \( \sum_{i=1}^{n} d(G)^i = \sum_{i=1}^{n} d(G')^i \).

**Proof.** By Lemma 2.1, \( G' \) is a connected graph with \( n \) vertices and \( n \) edges. So, \( G' \) is a unicyclic graph which contains a q-cycle, where \( q \) is the number of spanning trees of \( G' \) (given by the Laplacian spectrum, Lemma 2.1). As a consequence, \( G \) and \( G' \) have the same number of triangles and we can apply Lemma 2.4 and (8) of Lemma 2.1, get \( \sum_{i=1}^{n} d(G)^i = \sum_{i=1}^{n} d(G')^i \).

Here, we use the symbol \( \Phi \) to denote a forest, it is the union of components each of which is a tree. And use the symbol \( p(\Phi) \) to denote the product of the number of vertices in the components of \( \Phi \).

**Lemma 3.2** The coefficients \( l_i \) of the polynomial \( P_{L(G)}(\mu) \) are given by the formula

\[
(-1)^i l_i = \sum p(\Phi) \quad (1 \leq i \leq n),
\]

where the summation is over all sub-forests \( \Phi \) of \( G \) which has \( i \) edges.

**Theorem 3.3** No two non-isomorphic graphs of the form \( H_s\{C_q,T(n_1,n_2,n_3)\} \) are L-cospectral, where \( q \) is even.

**Proof.** Suppose that \( G' = H_s\{C_q',T(m_1',m_2',m_3')\} \) is L-cospectral to \( G = H_s\{C_q,T(m_1,m_2,m_3)\} \), by Lemma 3.1, \( q' = q \). Then,

\[
n_1 + n_2 + n_3 = n_1' + n_2' + n_3' \quad (3.1)
\]

By Lemma 3.2, we get

\[
(-1)^{r-2} l_{n-2} = q \sum_{i=0}^{n-1} (q + n_2 + n_3 + i)(n_1 - i) + q \sum_{i=0}^{n-1} (q + n_1 + n_3 + i)(n_2 - i) + q \sum_{i=0}^{n-1} (q + n_1 + n_2 + i)(n_3 - i) + \sum p(\Phi) \quad (3.2)
\]

where \( \Phi \) is over all sub-forests of \( H_s\{C_q,T(n_1,n_2,n_3)\} \) with \( n - 2 \) edges obtained by deleting two edges both from \( C_q \).

\[
\sum_{i=0}^{n-1} (q + n_2 + n_3 + i)(n_1 - i) = \sum_{i=0}^{n-1} (q + n_2 + n_3)n_1 + \sum_{i=0}^{n-1} (n_1 - q - n_2 - n_3)i - \sum_{i=0}^{n-1} i^2
\]

\[
= \frac{1}{2} q n_1^2 + \frac{1}{2} q n_2^2 + \frac{1}{2} q n_3^2 + \frac{1}{6} n_1^3 + \frac{1}{2} q n_1 \quad (\text{by Lemma 2.4 and (8) of Lemma 2.1})
\]
Also, we can get the formula about \( n_2 \) and \( n_3 \). Substituting the above formula about \( n_1 \), \( n_2 \) and \( n_3 \) into (3.2), we get

\[
(-1)^{r-2} l_{n-2} = q\left(\frac{1}{6}(n_1 + n_2 + n_3)^3 - n_1 n_2 n_3 + \left(\frac{1}{2} - \frac{1}{6}\right)(n_1 + n_2 + n_3) + \frac{1}{2} q(n_1 + n_2 + n_3)^2 + (1 - q)(n_1 n_2 + n_2 n_3 + n_3 n_1)\right) + \sum p(\Phi).
\]

For \( l_{n-2} \), we have the similar formula. Since \( n_1 + n_2 + n_3 = n_1' + n_2' + n_3' \), we have

\[
\sum p(\Phi) = \sum p(\Phi').
\]

Then \( l_{n-2} = \tilde{l}_{n-2} \) implies that

\[
(1 - q)(n_1 n_2 + n_2 n_3 + n_3 n_1) - n_1 n_2 n_3 = (1 - q)(n_1' n_2' + n_2' n_3' + n_3' n_1') - n_1' n_2' n_3'. \tag{3.3}
\]

If \( p \) even, \( G \) and \( G' \) are bipartite graph. By Lemma 2.5, their line graphs \( \Gamma(G) \) (see Figure.1) and \( \Gamma(G') \) are \( A \)-cospectral, i.e., \( \Gamma(G) \) and \( \Gamma(G') \) have the same adjacency characteristic polynomial. So, \( \phi(A(\Gamma(G)), 2) = \phi(A(\Gamma(G')), 2) \).

Now we use Lemma 2.2 to compute the adjacency characteristic polynomial of \( \Gamma(G) \). Delete \( v_i \) from \( \Gamma(G) \), we get a path \( P_{n_i - 1} \) and \( H_1 \) (see Figure. 2). There are five vertices which

![Figure 1. The line graph of G](image1)

are adjacent to \( v_1 \).

1. Deleting \( v_1 \) and \( v_2 \) from \( \Gamma(G) \) leaves \( P_{n_1 - 1} \), \( P_{n_2 - 1} \) and \( H_2 \) (see Figure. 3).
2. Deleting \( v_1 \) and \( v_3 \) leaves \( P_{n_1 - 1} \), \( P_{n_3 - 1} \) and \( H_3 \) (see Figure. 4).
3. Deleting \( v_1 \) and \( v_4 \) leaves \( P_{n_1 - 1} \) and \( H_4 \) (see Figure. 5).
4. Deleting \( v_1 \) and \( v_5 \) leaves \( P_{n_1 - 1} \) and \( H_5 \) (see Figure. 6).
5. Deleting \( v_1 \) and the vertex that is adjacent to \( v_1 \) on the path \( P_{n_1 - 1} \) leaves a path \( P_{n_2 - 2} \) and the graph \( H_1 \) (see Figure. 2).

![Figure 3. H_2](image2)

![Figure 4. H_3](image3)
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We partition the cycles containing $v_1$ into two sets A and B. Every cycle in set A does not contain the vertices on $C_q$ other than $v_1$ and $v_5$, but every cycle in set B does. First, we consider the cycles in set A. Since every cycle in set A contains the vertex $v_1$, we can construct the cycles in set A by choosing two, three, or four vertices from $v_2, v_3, v_4$ and $v_5$. We have the following cases.

**case 1.** Choose two vertices from $v_2, v_3, v_4$ and $v_5$. We have $\binom{4}{2}$ choices. So there are 6 distinct cycles, they are $v_1v_2v_3v_1$, $v_1v_2v_4v_1$, $v_1v_2v_5v_1$, $v_1v_3v_4v_1$, $v_1v_3v_5v_1$ and $v_1v_4v_5v_1$, respectively.

Delete $v_1, v_2$ and $v_3$ from $\Gamma(G)$. We get three paths and one cycle: $P_{n_1-1}, P_{n_2-1}, P_{n_3-1}$ and $C_q$.

Delete $v_1, v_2$ and $v_4$ from $\Gamma(G)$. We get three paths: $P_{n_1-1}, P_{n_2-1}$ and $P_{q+n_3-1}$.

Delete $v_1, v_2$ and $v_3$ from $\Gamma(G)$. We get three paths: $P_{n_1-1}, P_{n_2-1}$ and $P_{q+n_3-1}$.

Delete $v_1, v_3$ and $v_4$ from $\Gamma(G)$. We get three paths: $P_{n_1-1}, P_{n_3-1}$ and $P_{q+n_2-1}$.

Delete $v_1, v_3$ and $v_5$ from $\Gamma(G)$. We get three paths: $P_{n_1-1}, P_{n_3-1}$ and $P_{q+n_2-1}$.

Delete $v_1, v_4$ and $v_5$ from $\Gamma(G)$. We get three paths: $P_{n_1-1}, P_{q-2}$ and $P_{q+n_3}$.

**case 2.** Choose three vertices from $v_2, v_3, v_4$ and $v_5$. There are $\binom{4}{3}$ choices and each choice has $3!$ permutations. So there are 24 different cycles of length 4.

Delete $v_1, v_2, v_3$ and $v_4$ from $\Gamma(G)$. We get four paths: $P_{n_1-1}, P_{n_2-1}, P_{n_3-1}$ and $P_{q-1}$.

Delete $v_1, v_2, v_3$ and $v_5$ from $\Gamma(G)$. We get four paths: $P_{n_1-1}, P_{n_2-1}, P_{n_3-1}$ and $P_{q-1}$.

Delete $v_1, v_2, v_4$ and $v_5$ from $\Gamma(G)$. We get four paths: $P_{n_1-1}, P_{n_2-1}, P_{n_3}$ and $P_{q-2}$.

Delete $v_1, v_3, v_4$ and $v_5$ from $\Gamma(G)$. We get four paths: $P_{n_1-1}, P_{n_2}, P_{n_3-1}$ and $P_{q-2}$.

**case 3.** Choose four vertices from $v_2, v_3, v_4$ and $v_5$. There is only one choice and it corresponds to $4!$ permutations. So there are 24 different cycles of length 5 containing $v_1$.

Delete $v_1, v_2, v_3, v_4$ and $v_5$ from $\Gamma(G)$. We get four paths: $P_{n_1-1}, P_{n_2-1}, P_{n_3-1}$ and $P_{q-2}$.

Now, we consider the cycles in set B. Because the cycles in set B all contain $v_1$, they must also contain $v_4$ and $v_5$. So every cycle in set B contains all the vertices on $C_q$.

**case 1.** Cycles consisting of the vertices on $C_q$ and $v_1$. There is only one cycle. Delete the vertices on $C_q$ and $v_1$ from $\Gamma(G)$. We get two paths: $P_{n_1-1}$ and $P_{n_2+n_3}$.

**case 2.** Cycles consisting of the vertices on $C_q, v_1$ and $v_2$. There are two different cycles because $v_1$ and $v_2$ have two permutations, i.e., $v_1v_2$ and $v_2v_1$. Delete the vertices on $C_q$, $v_1$ and

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**Figure 5.** $H_4$

**Figure 6.** $H_5$
We get three paths: \( P_{m1 - 1}, P_{m2 - 1} \) and \( P_{m3} \).

**Case 3.** Cycles consisting of the vertices on \( C_q, v_1 \) and \( v_3 \). There are two different cycles because \( v_1 \) and \( v_3 \) have two permutations. Delete the vertices on \( C_q, v_1 \) and \( v_3 \). We get three paths: \( P_{m1 - 1}, P_{m2} \) and \( P_{m3 - 1} \).

**Case 4.** Cycles consisting of the vertices on \( C_q, v_1, v_2 \) and \( v_3 \). There are six distinct cycles because \( v_1, v_2 \) and \( v_3 \) have six permutations.

Let

\[
q_1 = 2\phi(A(G - v_1), 2) = 2p_{m1 - 1}(2)\phi(A(H_1), 2) = 2n_1\phi(A(H_1), 2). \quad (3.4)
\]

\[
q_2 = \sum_{u \in \mathcal{N}(v_1)} \phi(A(G - u - v_1), 2) = n_1n_2\phi(A(H_2), 2) + n_1n_3\phi(A(H_3), 2) + n_1\phi(A(H_4), 2)
+ n_1\phi(A(H_4), 2) + (n_1 - 1)\phi(A(H_4), 2). \quad (3.5)
\]

\[
q_3 = \sum_{Z \subseteq C(v_3)} \phi(A(G \setminus V(Z)), 2) = 0 + mn\phi(q + n_3) + mn\phi(q + n_3) + mn\phi(q + n_3)
+ m(n_3 + 1)(q - 1) + 6(n_3n_3q + mn\phi(q + n_3) + mn\phi(q + n_3) + mn\phi(q + n_3) + mn\phi(q + n_3)
+ 24mn\phi(q + n_3) + n_1(n_2 + n_3 + 1) + 2n_1n_3n_3 + 2n_1n_2n_3 + 6n_1n_2n_3
= 48n_1n_2n_3q - 22n_1n_2n_3 + 9n_1n_2q + 9n_1n_3q - 4n_1n_2 - 4n_1n_3 + n_1q. \quad (3.6)
\]

Then, by Lemma 2.2, we have

\[
\phi(A(G), 2) = q_1 - q_2 - 2q_3. \quad (3.7)
\]

Similarly, by Lemma 2.2 and Lemma 2.3, we have

\[
\phi(A(H_1), 2) = 2\phi(A(H_2 - v_3), 2) - \sum_{u \in \mathcal{N}(v_3)} \phi(A(H_2 - u - v_3), 2) - 2\sum_{Z \subseteq C(v_3)} \phi(A(H_2 \setminus V(Z)), 2)
= 2q + n_3 - (n_3q + (n_3 + 1)(q - 1)) + (q + n_3 - 1) - 2(n_3q - 1) + (n_3 + 1) + n_3
= -4n_2q \quad (3.8)
\]

\[
\phi(A(H_1), 2) = -4n_2q. \quad (3.9)
\]

\[
\phi(A(H_1), 2) = \phi(A(H_3), 2) = 2\phi(A(H_4 - v_2), 2) - \sum_{u \in \mathcal{N}(v_2)} \phi(A(H_4 - u - v_2), 2)
- 2\sum_{Z \subseteq C(v_2)} \phi(A(H_4 \setminus V(Z)), 2) = 4n_2n_3 - 4n_2n_q + n_2 + n_2 + n_3. \quad (3.10)
\]
\[
\phi(A(H_1),2) = 2\phi(A(H_1 - v_2),2) - \sum_{u \in N(v_2)} \phi(A(H_1 - u - v_2),2)
\]

\[
-2 \sum_{z \in V(Z)} \phi(A(H_1 \setminus V(Z)),2) = -22n_2n_3q - 4n_2q + 6n_2n_3 - 4n_3q
\]

(3.11)

Substitute (3.8),(3.9),(3.10) and (3.11) into (3.4) and (3.5), we have

\[
q_1 = 2n_q(-22n_2n_3q - 4n_2q + 6n_2n_3 - 4n_3q)
= -44n_2n_3q - 8n_2n_3q + 12n_2n_3 - 8n_2n_3q
\]

(3.12)

\[
q_2 = n_1n_2\phi(A(H_2),2) + n_1n_3\phi(A(H_3),2) + n_1\phi(A(H_4),2)
+ n_1\phi(A(H_5),2) + (n_1 - 1)\phi(A(H_6),2)
\]

\[
-38n_1n_2n_3q + 14n_1n_2n_3q + 2n_1n_3 + 2n_q + 2n_1n_3 - 4n_1n_2q
- 4n_1n_3q + 22n_1n_3q + 4n_2q - 6n_1n_3 + 4n_q
\]

(3.13)

Substitute (3.12),(3.13) and (3.6) into (3.7), we have,

\[
\phi(A(G),2) = q_1 - q_2 - 2q_3 = -102n_1n_2n_3q - 22n_1n_3q + 42n_1n_2n_3 - 22n_1n_3q + 6n_1n_2q
- 4n_3q + 6n_1n_3q - 22n_1n_3q - 4n_2q + 6n_2n_3 - 4n_3q
\]

(3.14)

Similarly, for \( G' \), we have

\[
\phi(A(G'),2) = -102n_1' n_2' n_3' q - 22n_1' n_2' n_3' q + 42n_1' n_2' n_3' q - 22n_1' n_3' q + 6n_1' n_2'
- 4n_1' q + 6n_1' n_3' - 22n_1' n_3' q - 4n_2' q + 6n_2' n_3' - 4n_3' q
\]

(3.15)

Since \( \phi(A(G),2) = \phi(A(G'),2) \), then,

\[
-102n_1n_2n_3q - 22n_1n_3q + 42n_1n_2n_3 - 22n_1n_3q + 6n_1n_2q + 6n_1n_3q - 22n_1n_3q
- 4n_2q + 6n_2n_3q - 4n_3q = -102n_1n_2n_3q - 22n_1n_3q + 42n_1n_2n_3 - 22n_1n_3q
+ 6n_1n_3q - 4n_1q + 6n_1n_3q - 22n_1n_3q - 4n_2q + 6n_2n_3q - 4n_3q
\]

(3.16)

Solve (3.1),(3.3) and (3.16) by using maple, we get the following result:

\[
\begin{align*}
&n_1 = n_1', n_2 = n_3', n_3 = n_2' \\
&n_1 = n_3', n_2 = n_1', n_3 = n_1' \\
&n_1 = n_1', n_2 = n_2', n_3 = n_3' \\
&n_1 = n_3', n_2 = n_2', n_3 = n_1' \\
&n_1 = n_1', n_2 = n_3', n_3 = n_1' \\
&n_1 = n_2', n_2 = n_1', n_3 = n_3' \\
&n_1 = n_1', n_2 = n_1', n_3 = n_2'
\end{align*}
\]

which implies that \( G \) is isomorphic to \( G' \), i.e., for \( q \) even, all the L-cospectral graphs \( H_1 \{C_n, T(n, n, n_3)\} \) are isomorphic.
4. Conclusion

In this paper, we firstly use the coefficient $l_{n-2}$ of Laplacian characteristic polynomial to obtain an equation. Then, we compute the adjacency characteristic polynomial of the corresponding line graph. For bipartite graphs which are L-cospectral, their line graphs are A-cospectral. Applying the Lemma, we proved that this class of L-cospectral graphs must be isomorphic.

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