Implementation of Sub-Nyquist Sampling System Based on Compressed Sensing

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Abstract

This paper describes the development of a sub-Nyquist sampling system that can digitize high-speed signals using a low-speed analog to digital converter (ADC). The system is implemented by a field programmable gate array (FPGA), and it is possible to make change to the equivalent sampling frequency according to the practical applications. As an application of the compressed sensing (CS) theory, for the spectrally sparse analog signal sampling, the proposed system has the potential to break though the constraint of Shannon theorem and the bandwidth barrier of state-of-the-art ADCs. Potential limitations of the applicability of CS-based sampling system are also discussed. Experimental results show that this sampling system is able to capture spectrally sparse analog signal at an equivalent sampling rate of 500 MHz while sampled at a rate of no more than 100 MHz physically.

Keywords: Compressed Sensing, Analog-To-Information Converter (AIC), Sparse Signal, Sub-Nyquist Sampling

1. Introduction

The Shannon sampling theorem provides the bridge between analog signal and discrete representation, allowing signal processing in a digital world. The theorem states that, without information loss, any signal with the highest frequency less than B Hz can be reconstructed from the samples captured at a rate of 2B Hz. The Shannon theorem lies at the heart of all analog to digital converters (ADCs). The improvement in state-of-the-art ADC is not sufficiently fast to catch up with the emerging applications in related fields, and many alternative sampling techniques have been proposed. Random equivalent sampling (RES) [1] forms a single waveform with a high equivalent sampling frequency after multiple acquisitions using a low-speed ADC, which employs the incoherence between the test signal and the sampling clock to capture a repetitive signal. While in practical sampling, due to incomplete randomness, it may not be guaranteed that the available samples are sufficient to reconstruct the underlying analog signal within desired accuracy. Time-interleaved sampling [2] presents an architectural possibility of improving the equivalent sampling frequency of data acquisition system, whose equivalent sampling frequency is proportional to the number of ADCs. Time-interleaved sampling could be cost effective or even only possible solution in sampling non-repetitive signal due to the high cost or unavailable of a single ADC solution. H. Olkkonen [3] proposed a sampling method based on parallel exponential filters, whose sampling efficiency is the same as that of time-interleaved sampling. Different from time-interleaved sampling, samples of all channels are taken using the same sampling clock, and it is not sensitive to timing uncertainty of sampling clock. However, all these sampling techniques are limited by the bandwidth barrier of state-of-the-art ADC even when prior information about the underlying analog signal is available.

In practical application, the information rate [4] of a continuous time signal is much lower than its highest frequency presented in the signal, and the Shannon sampling theorem is sufficient but not necessary. Recently, compressed sensing (CS) [5, 6] has been proposed as an efficient signal reconstruction method for random samples of sparse signals and obtains much attention [7, 8].. Some important theoretical works [9, 10] have been developed that demonstrate the feasibility of applying CS to sample spectrally sparse analog signals. An analog
signal is spectrally sparse if it consists of very few harmonics. For this family of analog signal, the CS theory ensures that very few random time samples will be sufficient to reconstruct the analog signal with great precision. Based on CS theory, S. Kirolos [11] developed an analog-to-information converter (AIC), which aims to sample periodic analog signal using ADC clocked at a rate much lower than the Nyquist rate. It has been proved that the AIC is an effective sampling system for implementing the random measurement procedure in the CS theory.

Inspired by AIC, we develop a sub-Nyquist sampling system with maximum equivalent sampling frequency of 500 MHz based on a field programmable gate array (FPGA) and a 100 MHz ADC. A pseudorandom sequence with the Nyquist frequency is generated by build-in random access memory (RAM) of FPGA, which is used to modulate the input spectrally sparse analog signal. The modulator spreads out the signal in frequency domain, which furnishes each frequency with a unique signature that can be discerned by examining the pass-band of the anti-aliasing filter. Experimental results show that the proposed system is effective and make it possible to sample and reconstruct high-speed periodic signal using low-frequency AD conversion device.

2. Theory and Application of CS

2.1 Theory basis

The theory of CS was developed by Ref. 5 and Ref. 6, which is a novel sampling paradigm that goes against the common wisdom in data acquisition. For a class of signals that exhibit a “spectral sparseness” property, CS method promises perfect reconstruction of the signal using very few random measurements. Sparsity expresses the idea that the information rate of a continuous time signal may be much smaller than its highest frequency presented in the signal, or that a discrete-time signal depends on a number of degrees of freedom that is comparably much smaller than its (finite) length. More precisely, CS exploits the fact that many man-made and natural signals are sparse or compressible in the sense that they have concise representations when expressed in an appropriate basis, such as Fourier basis, wavelet basis, etc. The basis can be selected according to the signal’s features.

In CS, the signal to be reconstructed is denoted by an $N$ dimensional vector $x$. In the current application, $x$ would be the unknown analog signal sampled at Nyquist rate over the time duration $T_e$. Using RES or other methods, a set of measurements of the elements of $x$ is obtained and denoted by a vector $y$. In RES, each element of $y$ is the output of a low-rate ADC. It can be represented as a weighted linear combination of elements in $x$ that contains evenly spaced samples of the unknown waveform, with the equivalent sampling rate $f_e > f_{Nyquist}$ ($f_{Nyquist}$ is the Nyquist rate of the signal to be measured). Let these weights be arranged in a measurement matrix $\Phi$, one may express the relation between $x$ and $y$ as follows:

$$ y = \Phi x. \quad (1) $$

For the periodic signal, it is sparse in Fourier domain. A signal is spectrally sparse if only very few Fourier coefficients have significant magnitudes while other Fourier coefficients are nearly zero. In other words, the energy of the signal is concentrated on few spectral coefficients. In particular, a $K$-sparse signal $x$ has $K$ significant spectral coefficients where $K$ is the sparsity level. For a $K$-sparse periodic signal $x(t)$, and its samples $x$ can be approximated by a linear combination of $K$ ($K << N$) discrete Fourier basis functions, i.e.,

$$ x = \sum_{i=1}^{K} \alpha_i \psi_{n_i} \quad (2) $$

where $\psi_{n_i} \in \Psi$, $n_i \in \{1, 2, ..., N\}$. For periodic analog signals, sparse representation basis $\Psi$ is the inverse discrete Fourier transform (IDFT) matrix [10]; for piecewise smooth signals, $\Psi$ is the inverse discrete wavelet transform (IDWT) matrix; for ultra wide band (UWB) pulse radio signals, $\Psi$ is the
unit matrix. Let \( \alpha = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T \) be the coefficients vector of \( x \) in \( \Psi \), and \( \alpha \) is a sparse vector consisting of \( K \) non-zero Fourier coefficients. Equation (3) can be rewritten as:
\[
y = \Phi x = D\alpha
\]
where \( D (= \Phi \Psi) \) is the equivalent measurement matrix. Given the observations \( y \), the measurement matrix \( \Phi \), and the transform matrix \( \Psi \), the purpose of CS reconstruction is to find \( \alpha \) such that
\[
||\alpha||_0 \text{ is minimized subject to } ||y - D\alpha||_2 \leq \eta.
\]
Here \( l_0 \) norm counts the number of non-zero elements in vector \( \alpha \), and \( \eta \) is the acceptable recovery error. Minimizing the \( l_0 \) norm of \( \alpha \) means to minimize the number of non-zero elements of the solution \( \tilde{\alpha} \) and therefore force the solution to be a sparse vector. Enforcing the constraint \( ||y - D\alpha||_2 \leq \eta \) would ensure the reconstructed signal \( x^\# = \Psi\tilde{\alpha} \) will yield (almost) the same measurements \( y \). There are many proposed algorithms to solve this minimization problem. Considering the computational complexity and difficulty of realization, greedy pursuit algorithm [12], especially orthogonal matching pursuit (OMP) [13], is attractive for engineering problems.

A sufficient condition for a sparse solution of \( \alpha \) to exist is that the matrix \( D \) must satisfy a restricted isometry property (RIP) [14]. Equivalently, the RIP condition can also be evaluated in terms of incoherence between the measurement matrix \( \Phi \) and the representation basis \( \Psi \) [15]. The coherence between \( \Phi \) and \( \Psi \) is defined as
\[
\mu(\Phi, \Psi) = \sqrt{N \cdot \max_{1 \leq i < j \leq N} \left| \langle \phi_i, \psi_j \rangle \right|}.
\]
To ensure a sparse solution \( \alpha^\# \) exists, \( \mu(\Phi, \Psi) \) should be as small as possible.

2.2 Sub-Nyquist sampling structure

AIC is significant result in the applications of CS theory in analog signal sampling, which offers a feasible approach to sampling signal at a rate that is proportional to “information rate” rather than the Nyquist rate. AIC consists of four parts: a pseudorandom sequence generator, a wideband demodulator, a low-pass filter (LPF), and low speed ADC, as shown in Fig. 1.

![Figure 1](image)

**Figure 1.** Block diagram of AIC. The demodulation process multiplies the input analog signal by a pseudorandom sequence with the Nyquist rate, which furnishes each frequency with a unique signature that can be discerned by examining the pass-band of the anti-aliasing filter.

Considering a periodic analog signal \( x(t) \), there exist continuous basis functions in Fourier basis such that
\[
x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \psi_n(t),
\]
with \( t, \alpha_n \in \mathbb{R}, \psi_n(t) \) is the column vector of Fourier basis. Since \( x(t) \) is sparse signal, there are a small number of non-zero entries in \( \alpha \). In order to find equivalent measurement matrix \( V = \Phi \Psi \), \( x(t) \) is fed to AIC, the output can be expressed as
\[
y[m] = \int_{-\infty}^{\infty} x(\tau) p_c(\tau) h(t - \tau) d\tau|_{mT_s}
\]
where $T_s$ is the sampling period of low-speed ADC and $m \in \{1, 2, \ldots, M\}$. Due to sparse representation of input signal, Eq. (7) can be revised as follow:

$$y[m] = \sum_{n=1}^{N} a_n \int_{-\infty}^{\infty} y_n(t) p_m(t) h(mT_s - t) \, dt.$$  

(8)

The element $v_{m,n} \in V$ for row $m$ and column $n$ can be separated out from Eq. (8)

$$v_{m,n} = \int_{-\infty}^{\infty} y_n(t) p_m(t) h(mT_s - t) \, dt.$$  

(9)

With the samples $y$ and the matrix $V$, the coefficients vector of input signal $x$ can be recovered with high probability, the signal approximation can be obtained by solving the inverse transform of Eq. (3).

The filter $h(t)$ can be any type in principle. In practice, $h(t)$ can be replaced by an integrator, and the integrator does not need to be high-fidelity. It suffices to perform low-pass filtering before the samples are taken.

### 3. System Design

#### 3.1 Hardware implementation

As mentioned in previous Section 2, we need to generate a pseudorandom sequence with the frequency no less than the Nyquist frequency to modulate the input signal. Generally, the shift-register devices are used to generate the pseudorandom sequence. For a sampling system based on the shift-register devices, it is difficult to make changes to the pseudorandom sequence on-board. On the other hand, In CS reconstruction, the length of the reconstructed signal should be the same as that of the pseudorandom sequence. However, most shift-registers can only generate less than 10-bits random numbers, for a long reconstructed signal, it will need large area of printed circuit board (PCB) to place shift-register devices. For example, if one wants to reconstruct a signal with length of 1000, there are about 100 pieces of shift-register device used to generate the pseudorandom sequence.

In this work, we develop a sampling system based on FPGA. With help of FPGA, we can reprogram the pseudorandom sequence and control its frequency on-board according to the practical applications. Fig. 2 shows a block diagram of sub-Nyquist sampling system, which can reconstruct spectrally sparse signal with equivalent sampling frequency up to 500 MHz from samples, which are taken using an ADC that is clocked at a rate of no more than 100 MHz.

The proposed system was implemented using off-the-shelf devices. The pseudorandom sequence generator is built in FPGA, and it is read out under control of a clock with maximum frequency of 500 MHz. FPGA is EP2S15, a member of Stratix II family from Altera [16]. The sampling clock and the pseudorandom sequence reading clock are generated by a build-in phase lock loop (PLL) of FPGA. Signal is firstly modulated with the pseudorandom sequence by a
mixer, out of which is then low-pass filtered and sampled uniformly using a low-speed ADC. The ADC AD9283-100 is with maximum sampling rate of 100 MHz and 8-bit resolution from Analog Devices [17]. The samples are stored in a first-in-first-out (FIFO) buffer memory in FPGA, which can be read out by a PCI interface. Then signal is reconstructed using CS reconstruction algorithm on a computer.

3.2 Implementation considerations

In our proposed system, the pseudorandom sequence is stored in a build-in RAM of FPGA. The stored random numbers are read out from RAM under control of a circular counter in FPGA, which are with i.i.d. Bernoulli distribution with value of “0” and “1”. The frequency of pseudorandom sequence is the same as that of the circular counter. We can change the pseudorandom sequence by programming FPGA according to the practical applications. A comparator is used to adjust the voltage level of the pseudorandom sequence to meet the input requirement of the mixer.

The cutoff frequency of LPF should be no more than $0.5f_s$ ($f_s$ is the sampling frequency) to avoid aliasing. In order to reconstruct a signal with length of $N$, we need to obtain $M$ consecutive samples, which should satisfy the following relation:

$$\frac{M}{f_s} = \frac{N}{f_r}.$$  (10)

Where $f_r$ is the equivalent sampling frequency of the reconstructed signal, and it also equals to the circular counter’s frequency. In CS, the number of samples required to ensure that $y$ retains all of information in $x$ is $M = O(K \log(N))$, and $K$ is the number of frequency components presented in the analog signal.

4. Experimental Results

Experimental waveforms were presented to investigate the performance of the proposed sampling system. The frequency of circular counter and the length of reconstructed signal were set to 500 MHz and 1000 respectively. In all the experiments, the bandwidth of LPF was set to $f_s/4$, and the sampling frequency $f_s$ is changed with $M$ (as described in (Eq. 10)).

The first test signal is a sinusoidal signal with 4 frequency components, whose highest frequency was 125 MHz. As introduced in Section 2.1, the sparse representation basis should be constructed by IDFT matrix. Obviously, the sampling rate of ADC is lower than the frequency of the test signal. According to Shannon sampling theorem, it is impossible reconstruct signal. In this work, we aim to employ CS theory to reconstruct the original signal from the 200 random sub-Nyquist samples. Fig. 3(a) shows the reconstruction, and the feasibility of the proposed method is clearly demonstrated. CS theory states that, one can require no more than $cK\log(N)$ ($c$ is a constant) random measurements to recover signal with overwhelming probability. Fig. 3(b) shows the reconstruction percentage with respect to the different number of samples $M$. The experimental results indict that, for a fixed $N$, the reconstruction performance is proportional to number of samples.
Figure 3. (a) Sinusoidal signal reconstruction, 100 reconstructed points are displayed. (b) Reconstruction percentage with respect to different numbers of samples. The length of reconstructed signal is $N = 1000$.

Sparsity is a fundamental premise in CS. For different types of signals, the sparse representation basis should be different. In this experiment, we consider two class signals: square-wave signal and UWB pulse signal. 200 random samples were used to reconstruct original signal. Fig. 4(a) shows the reconstruction of square-wave signal. For square-wave signal, there are infinite number of non-zero Fourier coefficients. Obviously, due to aggressive truncation of the Fourier series, the so called Gibb's phenomenon caused large overshoots of the CS reconstructed signal near the transition edges of the square wave. Fig. 4(b) shows the reconstruction of UWB pulse signal. As introduced in Section 2.1, UWB signal is sparse in time domain, and the unite matrix was employed in signal reconstruction. From Fig. 4, we notice that it is an issue to choose an appropriate representation basis for the specific signal.
5. Conclusion

As a new signal processing framework, CS aims to sample and reconstruct sparse signal at the rate that is much smaller than the Nyquist rate. AIC has been proved to be an effective application of CS theory. In this work, we developed a reconfigurable sub-Nyquist sampling system for sparse analog signal based on AIC. The system was implemented by a FPGA and a low-speed ADC, which enables to sample the high-speed sparse signals at the low sampling rate physically. With the help of FPGA, we can control the length and the frequency of the pseudorandom sequence without making changes to the hardware of system. We tested the developed sampling system with the experimental waveforms. With the help of CS theory, we can reconstruct the signal with high equivalent sampling frequency of 500 MHz from a small number of measurements. The results show that the proposed system is effective and practical for sparse signal processing.
number of samples captured using a low-speed ADC that is clocked at a rate of no more than 100 MHz. We also investigated the applicable limitation of the system. The performance of the proposed system directly depends on the sparsity level of the test signal, and the proposed sampling method can be used as a complement to the traditional sampling.

6. References


