A Class of Economic and Financial System’S Lyapunov Index and Its Application under the Condition of Parameters Bifurcation

Junhai Ma, Ying Wang

1College of Management Economic, Tianjin University, Tianjin 300072, China
2Tianjin University of Finance and Economics, Tianjin 300222, China
*1,2mjhtju@yahoo.com.cn, lzsly@126.com

Abstract
Based on the research work of scholars at home and abroad, this paper studies the inner complexity of the nonlinear model caused by the changes of three state variable interest rate, investment and price index, and have a further study of a series of characteristic of the system under combine station of nonlinear economic and financial system which contains information feedback loop as a key structure includes: a stable node, saddle point, bifurcation and Hopf bifurcation situation etc. Give the mathematical analytical and the Lyapunov index of the system under the condition of parameter combination. Then have the numerical simulation to the characteristics. For example economic operation stagnation, stable or not stable growth, inflation, and economic depression or economic situation out of control and so on, the numerical simulation results demonstrate the validity of the theoretical analysis, the results provide reference for this kind of economic and financial system and develop a way to the research of practical problems.

Keywords: Nonlinear Financial System, Stable Node, Saddle Point, Hopf Bifurcation, Lyapunov Exponent Impression

1. Introduction:

With the development of social and continuous improvement in human activities, people are facing with a more complex economic and financial system. Various factors such as mutual coupling, social interaction and mutual evolution has brought about into actual research. The scholars have committed to the field of complexity to control[1-4] and manage economic and financial systems which have started facing catastrophes since 1958. It has been found that, the theoretical and practical non-linear problems of economics arises a tremendous impact on economic and financial systems and thus creates chaos. Because the economic system appears non-linear chaos phenomena shows that the macroeconomic equilibrium have undergone a qualitative change which is controlled by long-term trends and other characteristics. While governments takes financial and or monetary policies and many other means to control real-time intervention, the effect is not immediate. Due to the complexity of the economic system’s accurate long-term forecast are very limited and short-term behavior is in the condition of complicated rational expectations[4-9], the effectiveness of such intervention and role cannot be overestimated to make the system balanced and under controlled.

Dragone D.[3] established an environmental externalities model with its stochastic optimal control to do the research; Fu SH, and others used the generalized Hamilton function system theory to study the synchronizention of chaotic systems; Su Z. et al[6] investigates a class of discrete non-linear with time delay power system stability and controllability. Therefore, they carried out an in depth study of these complex characteristics of the internal structure of the economic by studying the changes in system parameters corresponding to a series of complex features such as: periodic solution of the instability, bifurcation, period-doubling bifurcation, the location of the bifurcation point value system into the chaos of different roads, etc., which reveals the complex nature of the phenomenon that occurs during the process. Leonov GA[5] respectively, studied the continuous dynamic systems and discrete dynamical stability of the controlled linear system. This trend analysis system evolutionary becomes an increasingly important method for forecasting and control provide to theoretical and practical basis[10-13].
2. System Model Analysis

In this System Analysis Model, numerous changes occur in different states. The Interest Rate changes occurs in set variable $x$, where as changes in Investment Demand occurs in variable $y$ and finally $z$ variable denoted as the price index. The information contained between the three states variables are the key structure of the feedback loop. $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ respectively.

The following discussion will focus on the structural problems of this new set of state variables and thus includes the main factors affecting the variable $X$ change which occur in the investment market of demand and supply and the structural price adjustment. We use three main arguments to illustrate the simplified model:

Where $a \geq 0$ as the amount of savings, $b \geq 0$ as a unit investment cost and $c \geq 0$ as elasticity of demand for the commodity respectively. 

Shows the chaotic financial system of differential equations model, model (1) :

$$\dot{x} = z + (y - a)x, \dot{y} = 1 - by - x^2, \dot{z} = -x - cz$$

Model (1) of the three equilibrium points are:

$$p = \left(0, \frac{1}{b}, 0\right), q_{\pm} = \left(\pm \sqrt{\frac{c - b - abc}{c}}, \frac{1 + ac}{c}, \frac{1}{c} \sqrt{\frac{c - b - abc}{c}}\right) p = \left(0, \frac{1}{b}, 0\right)$$

The following research $c-b-abc$ illustrates the equilibrium point in the system $p = \left(0, \frac{1}{b}, 0\right)$. 

Let: $X=x, Y=y-\frac{1}{b}, Z=z$

Then (1) becomes:

$$\dot{X} = \frac{1}{b} X + Z + XY$$
$$\dot{Y} = -bY - X^2$$
$$\dot{Z} = -X - cZ$$

$p = (0, 0, 0)$ balance

The first case:

$$\begin{align*}
X &= XY + \frac{1}{b} X + Z \\
Y &= -bY - X^2
\end{align*}$$

Let: $\lambda_1 = -b, \lambda_2 = \frac{1}{b} - a + Y$

1.1. when $\lambda_1 = -b < 0, \lambda_2 = \frac{1}{b} - a + Y < 0, Y < a - \frac{1}{b}$, the system is stable.

1.2. When $\lambda_1 = -b < 0, \lambda_2 = \frac{1}{b} - a + Y > 0, Y > a - \frac{1}{b}$, the system is at a Saddle point.

The second case:

$$\begin{align*}
X &= XY + \frac{1}{b} X + Z \\
Z &= -X - cZ
\end{align*}$$
2.1. When $\lambda_1 > 0, \lambda_2 > 0$:
\[-(c + a - \frac{1}{b} - Y) > 0 \Rightarrow Y > c + a - \frac{1}{b}\]
\[1 + c(a - \frac{1}{b} - Y) > 0 \Rightarrow Y < a - \frac{1}{b} + \frac{1}{c}\]

2.2. When $\lambda_1 > 0, \lambda_2 < 0$:
\[-(c + a - \frac{1}{b} - Y) < 0 \Rightarrow Y < c + a - \frac{1}{b}\]
\[1 + c(a - \frac{1}{b} - Y) > 0 \Rightarrow Y < a - \frac{1}{b} + \frac{1}{c}\]

2.3. When $\lambda_1 < 0, \lambda_2 < 0$:
\[-(c + a - \frac{1}{b} - Y) < 0 \Rightarrow Y < c + a - \frac{1}{b}\]
\[1 + c(a - \frac{1}{b} - Y) > 0 \Rightarrow Y < a - \frac{1}{b} + \frac{1}{c}\]

2.4. When $\lambda_{1,2} = \alpha \pm i\beta, \alpha < 0$:
\[-(c + a - \frac{1}{b} - Y) < 0 \Rightarrow Y < c + a - \frac{1}{b}\]
\[(c + a - \frac{1}{b} - Y)^2 - 4[1 + c(a - \frac{1}{b} - Y)] < 0 \Rightarrow a - \frac{1}{b} - c - 2 < Y < a - \frac{1}{b} - c + 2\]

2.5. When $\lambda_{1,2} = \alpha \pm i\beta, \alpha > 0$:
\[-(c + a - \frac{1}{b} - Y) > 0 \Rightarrow Y > c + a - \frac{1}{b}\]
\[(c + a - \frac{1}{b} - Y)^2 - 4[1 + c(a - \frac{1}{b} - Y)] < 0 \Rightarrow a - \frac{1}{b} - c - 2 < Y < a - \frac{1}{b} - c + 2\]

2.6. When $\lambda_{1,2} = \pm i\beta$:
\[1 + c(a - \frac{1}{b} - Y) = i\beta(-i\beta) = -i^2\beta^2 = \beta^2 > 0 \Rightarrow Y < a - \frac{1}{b} + \frac{1}{c}\]
\[(c + a - \frac{1}{b} - Y)^2 - 4[1 + c(a - \frac{1}{b} - Y)] < 0 \Rightarrow a - \frac{1}{b} - c - 2 < Y < a - \frac{1}{b} - c + 2\]

2.7. When $\lambda_{1,2} = -2\alpha, \alpha < 0$:
\[-(c + a - \frac{1}{b} - Y) < 0 \Rightarrow Y < c + a - \frac{1}{b}\]

The Third case:
\[
\begin{cases}
Y = -bY - X^2 \\
Z = -X - cZ
\end{cases}
\tag{4}
\]

when $\lambda_1 = -b < 0, \lambda_2 = -c < 0$, the system is stable

3. Numerical simulation results:

Take the theoretical analyses into practical application and we can conclude the result below: and then the first 3000 points of 15,288 can be removed before the transient point.

When $a = 5, b = 0.2, c = 0.8$ and $a = 5, b = 0.16, c = 0.8, \Delta t = 0.01$ the txy variables is shown in figure 1 and figure 2 (Initial point $(x_0, y_0, z_0) = (0.03, 0.15, 0.25)$)
The numerical results show different combinations of the parameters make the system get into the state of a variety of complex phenomena, the value of deconstruction verify the theoretical correctness of the structure.

4. The Lyapunov index of the system and its complexity analysis

Consider the continuous-time nonlinear dynamical system

\[
x = Ax + N(x), \quad A = (a_{ij})_{n \times n}, \quad x \in \mathbb{R}^n
\]

Where \( N(x) = O(\|x\|^n) \) is a smooth function. Let the function \( N(x) \) be written as

\[
N(x) = \frac{1}{2} B(x, x) + \frac{1}{6} C(x, x, x) + O(\|x\|^4)
\]

Where \( B(x, y) \) and \( C(x, y, z) \) are bilinear and trilinear functions, respectively. In coordinates, we have

\[
B_i(x, y) = \sum_{j=1}^{3} \frac{\partial^2 N_i(x, \xi)}{\partial \xi_j \partial \xi_k} x_j y_k, \quad C_i(x, y, z) = \sum_{j, k=1}^{3} \frac{\partial^2 N_i(x, \xi)}{\partial \xi_j \partial \xi_k} x_j y_k z_k
\]

Suppose that \( A \) has a pair of complex eigenvalues on the imaginary axis: \( \lambda_{i, j} = \pm i \omega \quad (\omega > 0) \), and these eigenvalues are the only eigenvalues with \( \text{Re} \lambda = 0 \). Let \( q \in \mathbb{C}^n \) be a complex eigenvector to \( \lambda_i = i \omega: \quad Aq = i \omega q, \quad \overline{Aq} = -i \omega \overline{q} \)

Introduce also the adjoint eigenvector \( p \in \mathbb{C}^n \) admitting the properties:

\[
A^T p = -i \omega p, \quad A^T \overline{p} = i \omega \overline{p}
\]

and satisfying the normalization \( \langle q, p \rangle = 1 \)

Then we can get the coefficient matrix:

\[
\begin{bmatrix}
1 - iw & R & 1 \\
-2R & -b - iw & 0 \\
-1 & 0 & -c - iw
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

For solving it which equal to zero:
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\[ f(w) = -2cR^2 + \frac{ibw}{c} - ibcw - 2iR^2w + bw^2 - \frac{w^3}{c} + cw^2 + iw^3 \]

\[ = iw^3 + (b + c - \frac{1}{c})w^2 + (\frac{b}{c} - bc - 2R^2)iw - 2cR^2 = 0 \]

\[ Aq = iwq, A\overline{q} = -iw\overline{q}, A^\dagger p = -iwp, A^\dagger \overline{p} = iv\overline{p}, \]

Then, we can get:

\[ \begin{bmatrix} \frac{1}{c} & -2R & -1 \\ \frac{1}{b} & R & -b \\ 1 & 0 & -c \end{bmatrix}, \overline{q} = \begin{bmatrix} -c + iw \\ 2R(c+ iw) \\ b - iw \end{bmatrix}, \overline{p} = \begin{bmatrix} c+iw \\ R(c+ iw) \\ b+iw \end{bmatrix} \]

When \( \begin{bmatrix} w = a \\ f(a) = 0 \end{bmatrix} \), Matrix A exist pure virtual roots \( \pm ia \)

\[ \begin{bmatrix} 2abc - 2c + 2b \\ -c - b \end{bmatrix} = a \\
\]

\[ ia^2 + (b + c - \frac{1}{c})a^2 + (\frac{b}{c} - bc - 2R^2)ia - 2cR^2 = 0 \]

Reduction get

\[ -a^2 + 2abc + 2bc + a^2bc - 2c^2 + a^2c^2 = -2a^2bc - 3ab + 2ac - a^3c + abc^2 = 0 \]

Get the results for:

\[ L_1(a, b, c) = \frac{1}{2\omega} \end{bmatrix} < p, B(q, q, \overline{q}) > -2 < p, B(q, A^\dagger B(q, \overline{q})) > + < p, B(\overline{q}, (2i\omega E - A)^\dagger B(q, q)) > \]

\[ (a+ic)^2(-ia+c)\sqrt{\frac{c-b-abc}{c}} - 2(-ia+c)(ia+c)^2(-2ia+2c+\sqrt{\frac{c-b-abc}{c}}) + \]

\[ (ia-c)(a-ic)(-ia+c)(2ia^2 + 2(abc + b - c) + a^2(-1 + 2ic + c^2) + ia(4abc + 5b - 4c - bc^2)) \]

\[ (a-ib)(4a^2 - i(abc + b - c)2ia^2(-1 +bc + c^2) + a(2abc + 3b - 2c - bc^2)) \]

\[ = l_1(a, b, c) + im_1(a, b, c) \]

Remember actually department of \( l_1(a, b, c), m_1(a, b, c) \) for imaginary part, Through calculation, get the expression is as follows:

\[ l_1(a, b, c) = \text{Re}[L_1(a, b, c)] \]

\[ = \{ (-a^2 + c^2) \} \{ 68a^2bc^2 + 6b(abc + b - c)c^2 + a^2b[4abc^2 + 46c^2 - 22c^4 + b(-92x + 22c^3)] + \\
abc(70b - 70c + 22c^4) + b^2(51 - 10c^2 + 5c^4) \} + a \{ 6c + 88b^2c + 20b^2c^2 + abc(-6 + 88bc) + b(11 - 122c^2 + 17c^4) \} - \frac{(abc + b - c)}{c^2} \}

\[ \left( -a^2 + 5a^2(b - c)c + bc^2 \right) \{ 16a^2c^2 + (abc + b - c)^2 \}

\[ \{ 4a^2 + 4a^2(1 + 4abc + 4bc - 6c^2 + b^2c^2 + c^4) + a \{ 4abc^2 + 4bc(-4 + c^2) - 4c^2(-2 + c^2) + 4abc(3b - 3c + c^3) + b^2(9 - 2c^2 + c^4) \} \} \cos \left[ \frac{1}{2} \text{Arg} \left[ \frac{abc + b - c}{c} \right] \right] \]
\[ +a \left( \frac{(abc + b - c)^2}{c^2} \right)^{1/4} \left( c^2 (5b + c) + a^2 (b + 5c) \right)(16a^4c^2 + (abc + b - c)^2 c^2 + 4a^4 (1 + 4abc + 4bc - 6c^2 + b^2 c^2 + c^4) + a^2 (4abc^2 + 4bc(-4 + c^2) - 4c^2 (-2 + c^2) + 4abc(3b - 3c + c^3) + b^2 (9 - 2c^2 + c^4)) \right) \]

\[ +4abc(3b - 3c + c^3) + b^2 \left( 9 - 2c^2 + c^4 \right) \] \[ \frac{1}{2} \left( \frac{abc + b - c}{c} \right) \right) | \sin \left[ \frac{1}{2} \left( \frac{abc + b - c}{c} \right) \right] \right) / \left( (a^2 + b^2) \right) \]

\[ (16a^4c^2 + (abc + b - c)^2 c^2 + 4a^2 (1 + 4abc + 4bc - 6c^2 + b^2 c^2 + c^4) + a^2 (4abc^2 + 4bc(-4 + c^2) - 4c^2 (-2 + c^2) + 4abc(3b - 3c + c^3) + b^2 (9 - 2c^2 + c^4)) \] \[ \sin \left[ \frac{1}{2} \left( \frac{abc + b - c}{c} \right) \right] \right) / ((a^2 + b^2) \right) \]

\[ (16a^4c^2 + (abc + b - c)^2 c^2 + 4a^2 (1 + 4abc + 4bc - 6c^2 + b^2 c^2 + c^4) + a^2 (4abc^2 + 4bc(-4 + c^2) - 4c^2 (-2 + c^2) + 4abc(3b - 3c + c^3) + b^2 (9 - 2c^2 + c^4)) \]

Case 1: Then we have the discussion about the result shown in (15). When a is fixed, \( b \in (0, 5), c \in (0, 2) \), the Lyapunov graph is shown in figure 3.

**Figure 3.** The Lyapunov exponent value when \( a = 5, b \in (0, 5), c \in (0, 2) \)

As shown in figure 3, when \( a = 5, b \in (0, 5), c \in (0, 2) \), \( l_1(b, c) > 0 \), the Lyapunov exponent value of the system is greater than zero, and the system is in chaos state at this time, and when \( l_1(b, c) < 0 \), shows that the Lyapunov exponent value of the system is less than zero, and the system is not in chaos state at this time.

Will figure 3 projection to \( c-l_1(c) \) plane, and we get the Lyapunov exponent in this plane. As shown in figure 4.
Case 2: When \( b \) is fixed, \( a \in (0,10), \), \( c \in (0, 2) \) the Lyapunov graph is shown in figure 7, when \( b = 2.5, a \in (0,10), c \in (0, 2), l_1(a,c) > 0 \), the Lyapunov exponent value of the system is greater than zero, and the system is in chaos state at this time, and when \( l_1(a,c) < 0 \), Shows that the Lyapunov exponent value of the system is less than zero, and the system is not in chaos state at this time.

Will Figure 7 projection to a \( l_1(a) \) plane, The Lyapunov value of the system \( l_1(a) \) is reduce with the increase of the value of \( a \), and less than 0,explain that the system is not in chaos state at this time, And the complexity is more and more weak. The Lyapunov value of the system \( l_1(a) \) is basic less than zero, explain that the system is not in chaos state at this time. As shown in figure 8.
Case 3: When \( c \) is fixed, \( a \in (0,10), b \in (0,5) \), the Lyapunov exponent value of the system is greater than zero, and the system is in chaos state at this time, and when \( l_1(a,b) < 0 \), shows that the Lyapunov exponent value of the system is less than zero, and the system is not in chaos state at this time. As shown in figure 11.

Will figure 11 projection to \( b - l_1(b) \) plane, The Lyapunov value of the system \( l_1(b) \) is reduce with the increase of the value of \( a \), and \( l_1(0.1899) = 0 \) explain that when \( b > 0.1899 \) the system is not in chaos state, and the complexity is more and more weak, as shown in figure 12.
5. Conclusion:

This paper examined a series of economic system under the condition of different combination of characteristics, and give the expressions of the Lyapunov index under it. Achieve the internal topology structure and complexity analysis.

The mathematical analytical of the system’s Lyapunov index has played a very important role to understand the complex characteristics of the system. Get the threshold value of the system with the changes of parameter through the numerical simulation, and have very good guidance function to practical problems.

It is concluded that, the complex economic and financial systems is impacted because of the change of parameters which is under the condition of different combination. The main factors includes a saddle point, bifurcation and Hopf bifurcation et.al. Wherein a is the savings amount, b represents a unit investment cost and c is the commodities demanded respectively.

We found that, an economic recession which is as a result of unstable growth or economic stagnation will lead to a high inflation what’s more, it will trigger a serious social unrest. For the simulate the economic stagnation system, in this paper, we set $c - b - a b c < 0$, $x$ as the system parameters of interest, $y$ as investment demand, and $z$ as the topology price index which may be a stable exchange and or unstable equilibrium point which will result to a Hopf bifurcation and separation of stable and unstable limit cycles.

Acknowledgement

1. Supported by National Natural Science Foundation of China 61273231.

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